# Unobserved Mechanism Design: Targeted Offers

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**Abstract**

A seller of an individual object facing buyers with identically and independently distributed private valuations can benefit from making a targeted offer to buyers who reveal that they have high valuations, but this is not an equilibrium if the seller's mechanism is never observed by buyers (uninformed), because the seller is expected to make a low offer if all buyers have low valuations. We show that if buyers have an independent probability of observing the mechanism (informed), targeted-offer mechanisms can be an equilibrium. Uninformed buyers with valuations below a threshold (uninterested) signal their disinterest in an offer and allow the seller to target a single take-it-or-leave-it offer to uninformed buyers with valuations above the threshold (interested) who signal their interests. If the probability of being uninformed or the threshold for signaling interest is low, the revenue gain from making an offer to uninterested uninformed buyers is outweighed by the incentive cost of attracting informed buyers with low valuations, who observe any deviation by the seller. The seller's revenue decreases with the probability of the probability of being uninformed and increases with the threshold for signaling interest.

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# 1 Introduction

In a standard auction environment with a seller who has 0 reservation value for a single indivisible object and $n$ buyers whose private valuations are independently distributed with the same function $F$ over the unit interval, consider a game where the seller moves first and commits to a mechanism that generates a single take-it-or-leave-it offer to a selected buyer. Suppose that the revenue function from making a take-it-or-leave-it offer to a randomly selected buyer is strictly concave with the maximum achieved at $r^*$. If the seller's mechanism is observed by all buyers, the equilibrium of this game is Myerson's (1981) optimal auction, with reserve price $r^*$. If it is unobserved by any buyer, in any equilibrium the seller's revenue is the same as making a take-it-or-leave-it offer to a randomly selected buyer at $r^*$. In the latter case, both the seller and buyers with valuations above $r^*$ would benefit if the seller targets the offer of $r^*$ to buyers whose valuations are above any threshold $\xi \leq r^*$, as the offer is more likely to be accepted. However, it is not an equilibrium for buyers whose valuations are just above $r^*$ to reveal their interest in receiving the targeted offer, because they anticipate that the seller's would want to make a low offer when all buyers reveal their valuations to be below $\xi$. Can targeted-offer mechanism be an equilibrium in the intermediate case of buyers having an independent probability of observing the seller's mechanism?

In our earlier paper Li and Peters (2021), we have introduced the idea of unobserved mechanism design in a standard environment where buyers all have identical independently distributed valuations but they have the same probability of being uninformed of the mechanism that the seller has committed to. In this game of imperfect informative, there exist uncommunicative equilibria where uninformed buyers babble. Any equilibrium outcome corresponds to an optimal equal priority auction, in which informed buyers with all valuations on a pooling interval receive the good with the same probability as all uninformed buyers receive a take-it-or-leave-it offer.

In this follow-up paper, we consider "communicative equilibria" where uninformed buyers send messages to the seller's mechanism that are informative of their valuations. Since uninformed buyers would not be able to observe the seller's deviations, their communication strategy must be a best response to how the seller's equilibrium mechanism responds to the messages. In other words, the equilibrium communication strategy needs to be incentive compatible, which is the counterpart of incentive compatibility with respect to truthful reporting of valuations for informed buyers in a direct mechanism. We establish a version of the revelation principle for the purpose of finding a communicative equilibrium. For any equilibrium of the unobserved mechanism design game where uninformed buyers use a given communicative strategy, there is a direct mechanism that maps of profile of reported valuations from informed buyers and messages from uninformed buyers to a selection among buyers and the corresponding offer, such that

(i) it is incentive compatible for informed buyers not only for truthful revelation of their valuations but also for not pretending to be uninformed;

(ii) it achieves the same revenue for the seller;

(iii) it maximizes the seller's revenue among all direct mechanisms, taking as given the communicative strategy of uninformed buyers;

(iv) the given communicative strategy of uninformed buyers is a best response.

Conditions (i) and (ii) are counterparts of the standard revelation principle that helps narrow the search for a communicative equilibrium to direct mechanisms. Conditions (iii) and (iv) are unique to our unobserved mechanism design framework. Since uninformed buyers do not observe deviations by the seller, the latter is free to exploit any information content of the communicative strategy of the former. The two conditions are therefore the best response requirement for the seller and for uninformed buyers respectively. In particular, the optimality requirement in condition (iii) is conditional on the given communicative strategy of uninformed buyers, while best responding by uninformed buyers in condition (iv) is limited by their lack of knowledge about the seller's equilibrium mechanism.[1] Together, the four conditions could be seen as incorporating the standard optimal mechanism design in a fixed-point problem.

We use the above revelation principle to provide a partial characterization of communicative equilibria of our observed mechanism design game under the assumption that the seller's

[1]Condition (iv) is absent from the revelation principle we have established in Li and Peters (2021), because only babbling equilibrium is considered there.

revenue from making a take-it-or-leave-it offer is strictly concave. Restricting communication strategies of uninformed buyers to countable message spaces, we show that equilibrium messages can be numbered consecutively starting from 0, such that each message $l$ generates $l$ offers, and the $l$ offers generated by message $l$ are the same as the lowest $l$ offers generated by message $l + 1$ and each of them is generated with the same probability. The proof starts with the claim that if two messages generate only one offer, not only they have to be the same but they are also generated with the same probability. In equilibrium there must be uninformed buyers who send the message that generates each offer and who obtain a payoff of zero; otherwise the seller could increase the revenue by raising the offer in response to the message, without affecting the incentives of informed buyers (condition (i) above) or being detected by uninformed buyers (condition (iii) above). If the two offers are different, then uninformed buyers who send the message that generates the higher offer should deviate and send the message that generates the lower offer. If the two offers are the same, but one is generated with a greater probability than the other, then all uninformed buyers with valuations strictly above the offer should choose the message that generates it with a greater probability. In either case, the communication strategy would not be a best response for uninformed buyers (condition (iv) above). By induction, for any two distinct equilibrium messages used by uninformed buyers, one of them generates the same offers with the same probabilities as the other message, but generates additional higher offers. We then use the concavity assumption on the revenue function to show that, if two offers are generated in some equilibrium, either by the same message or by two messages, then the seller could pull them marginally closer to increase the revenue without violating the incentive conditions that informed buyers do not wish to pretend they are uninformed (condition (i) above). This cannot be an equilibrium because uninformed buyers who do not observe the seller's mechanism (condition (iii) above).

Since our revelation principle is conditional on a given communication strategy of uninformed buyers, and correspondingly, conditions (iii) and (iv) require a fixed-point argument, finding a communicative equilibrium of our unobserved mechanism design game does not reduce to a constrained maximization problem. We present a reverse of the above revelation principle: For any communication strategy of uninformed buyers, if there is a direct mechanism such that conditions (i), (iii) and (iv) above hold, then there is an equilibrium of the unobserved mechanism design game with the same payoff for the seller where uninformed buyers use the communication strategy. This "implementation" of a direct mechanism that satisfies conditions (i), (iii) and (iv) as an equilibrium exploits randomization by the seller to prevent uninformed buyers from behaving like informed buyers.

Now we combine the partial characterization of communicative equilibria and the implementation of direct mechanisms to construct a constrained optimization problem. We focus on the simplest class of equilibria with two messages, one of them generating one offer from the equilibrium mechanism and the other generating no offers. There is a threshold valuation $\xi$ such that uninformed buyers with valuations above the threshold send the message that generates an offer while those with valuations below the threshold send the message that generates no offers. We refer to the offer-generating message as "interested," and the other message as "uninterested." For a fixed threshold $\xi$, if we solve the seller's optimal direct mechanism and find that it involves making an offer to interested buyers higher than the threshold and making no offers to uninterested buyers, then conditions (i), (iii), (iv) of the revelation principle are all satisfied. By the implementation of direct mechanisms, we have found an equilibrium of the unobserved mechanism game.

Our main result is that when the probability that a given buyer is uninformed or the threshold $\xi$ for signaling interest in buying is small, the seller's best response is a direct mechanism that combines an equal priority auction where informed buyers with all valuations on a pooling interval receive the good with the same probability as all interested uninformed buyers receive a take-it-or-leave-it offer, with no offers to uninterested uninformed buyers. Intuitively, any revenue from making an offer to uninterested buyers is too small relative to the incentive cost of inviting informed buyers with low valuations to pretend to be uninterested uninformed buyers. If all buyers are known to be uninformed, the seller would not be able to commit to not making an offer to uninterested buyers when there is no one interested. Thus, it is the presence of informed buyers who discipline the seller and give the seller necessary credibility to make targeted offers.

The class of threshold equilibria constructed in this paper includes the babbling equilibrium constructed in Li and Peters (2021) as a special case. The seller's revenue is decreasing in the probability that a given buyer is uninformed, as the increasing presence of uninformed buyers erodes the effectiveness of the seller's mechanism. Furthermore, within the class of threshold equilibria, an increase in the threshold increases the seller's revenue, because a higher threshold allows the seller to better target the take-it-or-leave-it offers to uninformed buyers with high valuations.

The paper is organized as follows. The unobserved mechanism game is introduced in Section 2. In Section 3, we present a revelation principle generalized to our framework. Section 4 provides a partial characterization of equilibrium communication by uninformed buyers. Our main result is contained in Section 5, where we establish the existence of a class of communicative equilibria with targeted offers. We conclude with some remarks in Section 6 about related literature and future work. All skipped proofs, unless mentioned otherwise, can be found in the appendix section.

## 2 The Model

The model is the same as in Li and Peters (2021), and is presented here for the present paper to be self-containing. There are $n$ potential buyers of a single object owned by a seller. Each buyer $i$ has a privately known valuation $v_i$ that is independently drawn from some distribution $F$ with a continuously differentiable density function $f$ on $[0, 1]$. The seller's reservation value for the object is zero. Both the buyers and the seller are assumed to be risk-neutral.

The seller moves first and commits to a mechanism. The message space is $\mathcal{M} = [0, 1]^2$, which embeds the support $[0, 1]$ of valuations and the support $[0, 1]$ of a randomization device, and is common knowledge among the buyers and the seller. Denote a typical message from buyer $i$ as $b_i$. A mechanism maps a profile of $n$ messages to a single take-it-or-leave-it offer. More precisely, for any profile of messages, the output of the mechanism is a profile of pairs $(p_i, q_i)$, representing the probability of making buyer $i$ an offer $q_i$ and the offer itself $p_i$, with $\sum_{i=1}^{n} q_i \leq 1$. We allow $p_i$ to be stochastic. A pure strategy of the seller is a mechanism $\gamma = \{\mathcal{M}, p_i, q_i\}_{i=1}^{n}$. Let $\Gamma$ be the set of all mechanisms. A mixed strategy $\psi$ of the seller is defined in the usual way.

With probability $\alpha \in (0,1)$, independent their valuation, each buyer privately draws the information type "uninformed," representing that they do not observe the mechanism already committed by the seller. With the residual probability $1-\alpha$, the buyer is "informed," meaning they observe the mechanism. For each buyer $i = 1, \ldots, n$, we denote uninformed type as $\tau_i = \mu$ and the informed type as $\tau_i = \epsilon$. A pure strategy $\sigma_i$ for each buyer $i$ is a function $\sigma_i : [0,1] \times \{\epsilon, \mu\} \times \Gamma \to \mathcal{M}$, satisfying the informational constraint

$$\sigma_i (v_i, \mu, \gamma) = \sigma_i (v_i, \mu, \gamma') = \sigma_i (v_i, \mu)$$

for all $\gamma, \gamma' \in \Gamma$. We also allow buyers to use mixed strategies.

We denote a perfect Bayesian equilibrium of the above unobserved mechanism design game as $(\psi, (\sigma_i(v_i, \tau_i, \gamma))_{i=1}^n)$. This is defined in the standard way. We focus on symmetric equilibria, where the seller's strategy space is restricted to the subset of $\Gamma$ that contains all symmetric mechanisms (which treat any two buyers who have sent the same message in the same way), and where all uninformed buyers use the same strategy $\sigma(\cdot, \mu)$ and all informed buyers use the same strategy $\sigma(\cdot, \epsilon, \cdot)$. We say a symmetric equilibrium $(\psi, (\sigma(\cdot, \mu), \sigma(\cdot, \epsilon, \cdot)))$ is "communicative" if there are at least two distinct messages in the support of $\sigma(w, \mu)$, and uninformed buyers with for some valuation $w$ strictly prefer one of them.[2] In Li and Peters (2021), we consider uncommunicative equilibria in which uninformed buyers babble by uniformly randomizing over $[0,1]$, or equivalently, by choosing a single message from $\mathcal{M}$. Here, we examine the possibility of communicative equilibria.

# 3 Direct Mechanisms

In Li and Peters (2021), we define direct mechanisms and show that a version of the revelation principle holds, so that any equilibrium outcome can be supported via an optimal direct mechanism, and conversely, an optimal mechanism characterizes an equilibrium outcome. In this section, we extend the revelation principle to communicative equilibria where uninformed

[2]This definition rules out equilibria where uninformed buyers use an "informative" strategy in the sense that $\{w : \sigma(w, \mu) = b\} \neq \{w : \sigma(w, \mu) = \tilde{b}\}$ for some $b, \tilde{b}$ in the support of $\sigma(\cdot, \mu)$, but the seller's equilibrium mechanism does not distinguish $b$ and $\tilde{b}$.

buyers use strategies that are informative of their valuations.

For our purpose, we will define direct mechanisms with respect to a fixed communication strategy of uninformed buyers, which we denote from now on as $\sigma^\mu$ for notational brevity. Let $\mathcal{M}^\mu \subseteq \mathcal{M}$ be the support of $\sigma^\mu$. For simplicity, we assume that $\sigma^\mu$ is a pure strategy; the case of mixed strategies is a straightforward extension. Denote as $m$ the number of uninformed buyers. Let $v^\epsilon$ be the profile of $n-m$ valuations of informed buyers, and let $b^\mu$ be the profile of $m$ messages of uninformed buyers. Reorder $n$ buyers such that the first $n-m$ of them are informed. For each $v=(v_1,\ldots,v_n)\in[0,1]^n$, and for each $i=1,\ldots,n-m$, let

$$\rho_i^\epsilon(v^\epsilon)=(v_i,v_2,\ldots,v_{i-1},v_1,v_{i+1},\ldots,v_{n-m},v_{n-m+1},\ldots,v_n\},$$

so that $\rho_i^\epsilon$ switches the positions of the first and $i$-th valuations for informed buyers. For each $j=n-m+1,\ldots,n$, let

$$\rho_j^\mu(b^\mu)=(b_n,b_{n-m+2},\ldots,b_{n-1},\ldots,b_j\},$$

so that $\rho_j^\epsilon$ switches the positions of the $(n-m+1)$-th and the last messages for uninformed buyers. A feasible symmetric direct mechanism $\delta$ is a collection of functions

$$\delta=\left\{(q_m^\epsilon(v^\epsilon;b^\mu),p_m^\epsilon(v^\epsilon;b^\mu))_{m=0}^{n-1},(q_m^\mu(v^\epsilon;b^\mu),p_m^\mu(v^\epsilon;b^\mu))_{m=1}^{n}\right\}$$

where $q_m^\tau(v^\epsilon;b^\mu)$ and $p_m^\tau(v^\epsilon;b^\mu)$ for each $\tau=\epsilon,\mu$ map $[0,1]^{n-m}\times(\mathcal{M}^\mu)^m\to[0,1]$, and satisfy

- $(q_m^\tau(v^\epsilon;b^\mu),p_m^\tau(v^\epsilon;b^\mu))$ for each $\tau=\epsilon,\mu$ is variant to permutations of $(v_2,\ldots,v_{n-m})$, and to permutations of $(b_{n-m+1},\ldots,b_{n-1})$;
- for all $v^\epsilon$ and $b^\mu$,

$$\sum_{i=1}^{n-m} q_m^\epsilon\left(\rho_i^\epsilon\left(v^\epsilon\right);b^\mu\right)+\sum_{j=n-m+1}^{n} q_m^\mu(v^\epsilon;\rho_j^\mu(b^\mu))\leq 1. \tag{1}$$

Fix a strategy of uninformed buyers $\sigma^\mu$, and let $\sigma^\mu(v^\mu)$ be the profile of messages from uninformed buyers when the profile of their valuations is $v^\mu$. Given any direct mechanism $\delta$, for each $m = 0, \ldots, n-1$ define

$$Q^{\epsilon}_{m}(w; \sigma^{\mu}) = \mathbb{E}_{v^{\epsilon}_{-1}; v^{\mu}} \left[ q^{\epsilon}_{m}(w, v^{\epsilon}_{-1}; \sigma^{\mu}(v^{\mu})) \right],$$

where $(w, v^{\epsilon}_{-1})$ is the profile of valuations of informed buyers, with $v_1 = w$. Similarly,

$$P^{\epsilon}_{m}(w; \sigma^{\mu}) = \mathbb{E}_{v^{\epsilon}_{-1}; v^{\mu}} \left[ q^{\epsilon}_{m}(w, v^{\epsilon}_{-1}; \sigma^{\mu}(v^{\mu})) p^{\epsilon}_{m}(w, v^{\epsilon}_{-1}; \sigma^{\mu}(v^{\mu})) \right].$$

Taking expectations over $m$, we define

$$Q^{\epsilon}(w; \sigma^{\mu}) = \sum_{m=0}^{n-1} B(m; n-1, \alpha) Q^{\epsilon}_{m}(w; \sigma^{\mu})$$
$$P^{\epsilon}(w; \sigma^{\mu}) = \sum_{m=0}^{n-1} B(m; n-1, \alpha) P^{\epsilon}_{m}(w; \sigma^{\mu}),$$

where

$$B(m; n-1, \alpha) = \begin{pmatrix} n-1 \\ m \end{pmatrix} (1-\alpha)^{n-1-m} \alpha^{m}$$

is the probability that $m$ out of $n-1$ buyers are uninformed. Let

$$U^{\epsilon}(w; \sigma^{\mu}) = Q^{\epsilon}(w; \sigma^{\mu}) w - P^{\epsilon}(w; \sigma^{\mu}).$$

Then, for fixed $\sigma^{\mu}$, we say that the mechanism $\delta$ is $\sigma^{\mu}$-incentive compatible for informed buyers with respect to valuations, if the payoff to an informed buyer with valuation $w$ can be written as

$$U^{\epsilon}(w; \sigma^{\mu}) = \int_{0}^{w} Q^{\epsilon}(x; \sigma^{\mu}) \, dx, \tag{2}$$

with $Q^{\epsilon}(w; \sigma^{\mu})$ non-decreasing in $w$.

The payoff $U^{\mu}(w, b; \sigma^{\mu})$ to an uninformed buyer with valuation $w$ from using any message $b \in \mathcal{M}^{\mu}$, when all other uninformed buyers use the strategy $\sigma^{\mu}$, is

$$\sum_{m=0}^{n-1} B(m; n-1, \alpha) \mathbb{E}_{v^{\epsilon}; v^{\mu}_{-n}} \left[ q^{\mu}_{m+1}(v^{\epsilon}; \sigma^{\mu}_{-n}(v^{\mu}_{-n}), b) \max \left\{ w - p^{\mu}_{m+1}(v^{\epsilon}; \sigma^{\mu}_{-n}(v^{\mu}_{-n}), b), 0 \right\} \right], \tag{3}$$

where $v^\mu_{-n}$ is the profile of valuations of other $m$ uninformed buyers, and $(\sigma^\mu_{-n}(v^\mu_{-n}), b)$ is the profile of messages from all other uninformed buyers and the present uninformed buyer with valuation $w$, with $b_n = b$.[3] This is the same payoff $U^\epsilon(w, b; \sigma^\mu)$ received by an informed buyer with valuation $w$ from using the same message $b$. We say that $\delta$ is $\sigma^\mu$-incentive compatible for uninformed buyers if for all $w$,

$$U^\mu\left(w, \sigma^\mu(w); \sigma^\mu\right) \geq U^\mu\left(w, b; \sigma^\mu\right) \tag{4}$$

for all $b \in \mathcal{M}^\mu$.

We say that $\delta$ is $\sigma^\mu$-incentive compatible for informed buyers if for all $w$, if $\delta$ is incentive compatible for informed buyers with respect to valuations and if

$$U^\epsilon(w; \sigma^\mu) \geq U^\mu\left(w, b; \sigma^\mu\right) \tag{5}$$

for all $b \in \mathcal{M}^\mu$. Finally, $\delta$ is $\sigma^\mu$-incentive compatible if it is $\sigma^\mu$-incentive compatible for both informed and uninformed buyers.

Define

$$\phi(w) = w - \frac{1 - F(w)}{f(w)}$$

as the virtual valuation of $w$. The seller's revenue from informed buyers under any incentive compatible mechanism is given by

$$\begin{aligned}
&\sum_{m=0}^{n-1} B(m; n, \alpha)\mathbb{E}_{v^\epsilon; v^\mu}\left[\sum_{i=1}^{n-m} p^\epsilon_m(\rho^\epsilon_i(v^\epsilon); \sigma^\mu(v^\mu))\right] \\
=&\sum_{m=0}^{n-1} B(m; n, \alpha)(n - m)\mathbb{E}_w\left[P^\epsilon_m(w; \sigma^\mu)\right] \\
=&n(1 - \alpha)\mathbb{E}_w\left[Q^\epsilon\left(w; \sigma^\mu\right) w - U^\epsilon\left(w; \sigma^\mu\right)\right] \\
=&n(1 - \alpha)\int_0^1 Q^\epsilon\left(w; \sigma^\mu\right)\phi(w)f(w)dw,
\end{aligned} \tag{6}$$

where the last step uses the incentive compatibility with respect to valuations for informed

[3]In (3), we have assumed that uninformed buyers are unable to act like informed buyers. This will be ensured later through randomization by the seller.

buyers and integration by parts, and then imposes $U^\epsilon(0;\sigma^\mu)=0$.

The seller's revenue from uninformed buyers is given by

$$\begin{aligned}
&\sum_{m=1}^{n} B(m;n,\alpha)\mathbb{E}_{v^\epsilon;v^\mu}\left[\sum_{j=n-m+1}^{n} q_m^\mu(v^\epsilon;\rho_j^\mu(\sigma^\mu(v^\mu)))p_m^\mu(v^\epsilon;\rho_j^\mu(\sigma^\mu(v^\mu)))\mathbb{1}_{v_j\geq p_m^\mu(v^\epsilon;\rho_j^\mu(\sigma^\mu(v^\mu)))}\right] \\
=&\sum_{m=1}^{n} B(m;n,\alpha)m\mathbb{E}_w\Big[\mathbb{E}_{v^\epsilon;v_{-n}^\mu}\big[q_m^\mu(v^\epsilon;\sigma_{-n}^\mu(v_{-n}^\mu),\sigma^\mu(w))\cdot \\
&\qquad p_m^\mu(v^\epsilon;\sigma_{-n}^\mu(v_{-n}^\mu),\sigma^\mu(w))\mathbb{1}_{w\geq p_m^\mu(v^\epsilon;\sigma_{-n}^\mu(v_{-n}^\mu),\sigma^\mu(w))}\big]\Big] \\
=&n\alpha\mathbb{E}_w\left[\sum_{m=0}^{n-1} B(m;n-1,\alpha)\mathbb{E}_{v^\epsilon;v_{-n}^\mu}\big[q_{m+1}^\mu(v^\epsilon;\sigma_{-n}^\mu(v_{-n}^\mu),\sigma^\mu(w))\cdot\right. \\
&\qquad \left.p_{m+1}^\mu(v^\epsilon;\sigma_{-n}^\mu(v_{-n}^\mu),\sigma^\mu(w))\mathbb{1}_{w\geq p_{m+1}^\mu(v^\epsilon;\sigma_{-n}^\mu(v_{-n}^\mu),\sigma^\mu(w))}\big]\right]. \qquad (7)
\end{aligned}$$

The seller's total revenue $R(\delta)$ from a direct mechanism $\delta$ when uninformed buyers use strategy $\sigma^\mu$ is the sum of (6) and (7).

We can now state a version of the revelation principle. The proof is straightforward, and therefore omitted.[4]

**Theorem 1** *Fix a strategy $\sigma^\mu$ of uninformed buyers. For any symmetric equilibrium in which uninformed buyers use $\sigma^\mu$, there is a feasible, $\sigma^\mu$-incentive compatible direct mechanism $\delta^*$ that achieves the equilibrium expected revenue for the seller and $R(\delta^*)\geq R(\delta)$ for every feasible, $\sigma^\mu$-incentive compatible direct mechanism $\delta$. Conversely, any feasible, $\sigma^\mu$-incentive compatible direct mechanism $\delta^*$ that maximizes $R(\delta)$ can be used to construct an equilibrium in which uninformed buyers use $\sigma^\mu$.*

There are two parts to Theorem 1. The first part, which "replicates" an equilibrium in the unobserved mechanism game through a direct mechanism $\delta^*$, mimics the standard revelation principle arguments for informed buyers; the only difference here is an additional one-sided incentive compatibility constraint (5), which requires informed buyers not to pretend to be uninformed. Unlike the standard revelation principle, however, we require that the replicating direct mechanism $\delta^*$ to be optimal for the seller, taking as given the commu-

[4]The formal arguments parallel the proof of a similar result in Li and Peters (2021). We refer to interested readers to our earlier paper.

nication strategy $\sigma^\mu$ of uninformed buyers. The idea is that the seller can deviate to another mechanism without being detected by uninformed buyers, and given that informed buyers observe such deviation, there is no loss in considering only direct mechanisms in a possible deviation. However, since uninformed buyers have equilibrium expectations regarding the seller responds to messages in $\mathcal{M}^\mu$, we also require condition (4), so that $\sigma^\mu$ to be a response to the replicating direct mechanism $\delta^*$. Taken together, the replication part of Theorem 1 formulates a symmetric equilibrium of the unobserved mechanism design game as a fixed point in a mapping from a communication strategy of uninformed buyers to a best response by uninformed buyers to the optimal direct mechanism of the seller.

The second part of Theorem 1 "implements" a fixed point in the best response mapping defined by the first part as an equilibrium of the the unobserved mechanism design game. This part is needed because after we construct a fixed point we still need to ensure that there is a way for the seller to prevent uninformed buyers from acting like informed buyers. We do so by allowing the seller to randomize among direct mechanisms that are identical except for how valuations of informed buyers are reported. One way of doing so has a straightforward interpretation: the seller introduces a password that is uniformly distributed on $[0,1]$, and reveals it to informed buyers whose messages are then taken as reports of their valuations, while uninformed buyers do not observe the password and thus prevented from acting like informed buyers.

# 4 Equilibrium Communication

In this section, we use the replication part of Theorem 1 to provide a partial characterization of communicative equilibria of our unobserved mechanism design game. Fix some $\sigma^\mu$. By Theorem 1, any equilibrium outcome is associated with a direct mechanism

$$\left\{ (q_m^\epsilon(v^\epsilon; b^\mu), p_m^\epsilon(v^\epsilon; b^\mu))_{m=0}^{n-1}, (q_m^\mu(v^\epsilon; b^\mu), p_m^\mu(v^\epsilon; b^\mu))_{m=1}^{n} \right\}.$$

From (3), we say that a message $b$ generates an offer $t$ if the probability $\chi$, defined below, is strictly positive:

$$\chi \equiv \sum_{m=0}^{n-1} B(m; n-1, \alpha)\mathbb{E}_{v^{\epsilon}; v^{\mu}_{-n}} \left[ q^{\mu}_{m+1}(v^{\epsilon}; \sigma^{\mu}_{-n}(v^{\mu}_{-n}), b) \left| p^{\mu}_{m+1}(v^{\epsilon}; \sigma^{\mu}_{-n}(v^{\mu}_{-n}), b) = t \right. \right]. \quad (8)$$

We consider only communicative equilibria in which each message $b$ in the support of $\sigma^{\mu}$ generates a countable number of offers.

Suppose that there are two messages, $b$ and $\tilde{b}$, that are in the support of $\sigma^{\mu}$ and each generate at least one offer. We first show that in any equilibrium the two offer distributions coincide for the lowest offers.

**Lemma 1** *Suppose that in some equilibrium $b \in \mathcal{M}^{\mu}$ generates offers $t^1 < \ldots < t^l$ with probabilities $\chi^1, \ldots, \chi^l$, and $\tilde{b} \in \mathcal{M}^{\mu}$ generates offers $\tilde{t}^1 < \ldots < \tilde{t}^{\tilde{l}}$ with probabilities $\tilde{\chi}^1, \ldots, \tilde{\chi}^{\tilde{l}}$. Then, $t^j = \tilde{t}^j$ and $\chi^j = \tilde{\chi}^j$ for each $j = 1, \ldots, \min\{l, \tilde{l}\}$.*

**Proof.** Consider $t^j$ for any $j = 1, \ldots, l$. We assume that $t^j$ is a serious offer, meaning that there exists valuation $w$ of an uninformed buyer $i$ such that for whom $\sigma^{\mu}(w) = b$ and $w > t^j$; otherwise, the seller could set $\chi^j = 0$ without affecting the revenue or violating the incentive condition (5) of informed buyers. Let

$$\beta(t^j) = \inf \left\{ w : \sigma^{\mu}(w) = b \text{ and } w > t^j \right\}.$$

By definition, we have $\beta(t^j) \geq t^j$. We claim that $\beta(t^j) = t^j$. If this were false, the seller could raise the revenue by increasing his price offer $t^j$ on a set of positive measure, contradicting the equilibrium condition imposed by Theorem 1 that the seller's mechanism is a best response to $\sigma^{\mu}$. The same argument establishes that for any $j = 1, \ldots, \tilde{l}$,

$$\tilde{\beta}(\tilde{t}^j) \equiv \inf \left\{ w : \sigma^{\mu}(w) = \tilde{b} \text{ and } w \geq \tilde{t}^j \right\} = \tilde{t}^j.$$

Now, compare $t^j$ and $\tilde{t}^j$, and $\chi^j$ and $\tilde{\chi}^j$. If $t^j > \tilde{t}^j$, since $t^j$ is the lowest offer generated by $b$, an uninformed buyer $i$ with valuation $\beta(t^j) = t^j$ receives an expected payoff of zero by sending message $b$, but by deviating and sending message $\tilde{b}$, he would receive a strictly positive payoff of $\tilde{\chi}^j(t^j - \tilde{t}^j)$. This contradicts the equilibrium condition imposed by Theorem 1 that the seller's equilibrium mechanism is $\sigma^\mu$-incentive compatible for uninformed buyers. Thus, $t^j = \tilde{t}^j$. If $\chi^j < \tilde{\chi}^j$, then given $t^j = \tilde{t}^j$, an uninformed buyer $i$ with any valuation strictly between $t^j$ and $\min\{t^{j+1}, \tilde{t}^{j+1}\}$ strictly prefers sending message $b$ to sending message $\tilde{b}$, contradicting the result that $\beta(t^j) = t^j$. Thus, $\chi^j = \tilde{\chi}^j$. By induction, $t^j = \tilde{t}^j$ and $\chi^j = \tilde{\chi}^j$ for each $j = 1, \ldots, \min\{l, \tilde{l}\}$. ■

Given the definition of $\chi$ in (8), by the seller's revenue from uninformed buyers (7), the total probability that the seller's equilibrium mechanism responds to an equilibrium message $b$ by uninformed buyers with an offer $t$ is

$$n\alpha\chi \int_{\{w|\sigma^\mu(w)=b\}} dF(w). \tag{9}$$

Define the seller's "conditional revenue function" from making offer $t$ in response to message $b$ as

$$\pi(t|b) \equiv t \int_{\{w \geq t|\sigma^\mu(w)=b\}} dF(w). \tag{10}$$

The "unconditional" revenue function, denoted as $\pi(\cdot)$, is given by

$$\pi(t) = t(1 - F(t)).$$

By Lemma 1, we can rank distinct equilibrium messages in increasing order by the number of offers they generate. Write the messages in $\mathcal{M}^\mu$ as $b^l$, $l = 0, 1, \ldots$. Message $b^0$ generates no offers. Message $b^{l+1}$ generates at least 1 more offer than $b^l$, and the former generates all offers generated by the latter, with the same probabilities. Our next characterization result uses the assumption that $\pi(\cdot)$ is strictly concave to show that $b^{l+1}$ generates exactly 1 more offer than $b^l$.

**Lemma 2** *Suppose $\pi(\cdot)$ is strictly concave. In any symmetric equilibrium with a countable number of offers generated by $\sigma^\mu$, there is an equal number of distinct equilibrium messages in the support of $\sigma^\mu$ that generate at least one offer, and each such message $b^l$ generates offers $t^1, \ldots, t^l$ with probabilities $\chi^1, \ldots, \chi^l$.*

**Proof.** Suppose that for some $l$, all messages $b^{l+1}$ and higher generate offers $t < \tilde{t}$, with probabilities $\chi$ and $\tilde{\chi}$ respectively, while all messages $b^l$ and lower do not generate either offer. Consider another direct mechanism by marginally increasing $t$ and marginally decreasing $\tilde{t}$ for all messages $b^{l+1}$ and higher, so that

$$-\chi dt - \tilde{\chi} d\tilde{t} = 0.$$

The expected payoff of an informed buyer from pretending to be uninformed is either unchanged, if his valuation $w$ is below $t$ or above $\tilde{t}$, or decreased, if his valuation is between $t$ and $\tilde{t}$ and sends message $t^{l+1}$ or above. It follows that the incentive compatibility of informed buyers remains satisfied.

Then, given how we have constructed $dt > 0 > d\tilde{t}$ to satisfy the incentive compatibility constraint of informed buyers, from (7), (8) and (9) we have that the total effects on the seller's expected revenue of marginally increasing $t$ and marginally decreasing $\tilde{t}$ with respect to any message $b^{l+1}$ or higher can be written as

$$n\alpha\chi \int_{\{w|\sigma^{\mu}(w)=b\}} dF(w) \left( \frac{d\pi(t|b)}{dt} - \frac{d\pi(\tilde{t}|b)}{dt} \right) dt.$$

Since by Lemma 1 messages $b^{l+1}$ and higher generate all offers generated by messages $b^l$ and lower with the same probabilities, and offers $t$ and $\tilde{t}$ are only generated by messages $b^{l+1}$ and higher, uninformed buyers with valuations above $t$ strictly prefer messages $b^{l+1}$ and higher to messages $b^l$ and lower. Summing over all messages $b^{l+1}$ and higher, we have the total effects as

$$n\alpha\chi \left( \frac{d\pi(t)}{dt} - \frac{d\pi(\tilde{t})}{d\tilde{t}} \right) dt.$$

Since $\pi(\cdot)$ is strictly concave, $t < \tilde{t}$ and $dt > 0$, the above is strictly positive. This contradicts the equilibrium condition from Theorem 1 that the seller's equilibrium mechanism is optimal given uninformed buyers' strategy $\sigma^{\mu}$. ■

Using Lemma 2, we have a partial characterization of equilibrium communication $\sigma^{\mu}$ by uninformed buyers. In equilibrium, in addition to a possible message $b^0$ that generates now offers, there is a countable number of distinct messages in $\mathcal{M}^{\mu}$. For each $l \geq 1$, uninformed buyers with valuation on each interval $(t^l, t^{l+1}]$ are indifferent among messages $b^l$ and higher, as they all generate offers $t^1, \ldots, t^l$ with the same probabilities. Thus, these uninformed buyers can randomize among all messages $b^l$ and higher. At the same time, they strictly prefer messages $b^l$ and higher to messages $b^{l-1}$ and lower.

# 5 Targeted Offers

In this section we will construct the simplest class of communicative equilibria consistent with our partial characterization of equilibrium communication by uninformed buyers in Section 4. There are only two messages used by uninformed buyers, one generating a single offer while the other generating no offers. We refer to these two messages as "interested," denoted as $b^1$, and "uninterested," denoted as $b^0$. Such equilibrium is necessarily informative, because uninformed buyers with valuations above the offer generated by $b^1$ strictly prefer $b^1$ to $b^0$. Since an interested uninformed weakly prefers $b^1$ to $b^0$, and an uninterested buyer is indifferent between the two messages, the incentive compatibility constraints for uninformed buyers (4) are always satisfied so long as those with valuations higher than the offer generated by $b^1$ send $b^1$.

For our main result in this section, we will further restrict to equilibria with $\sigma^\mu$ by uninformed buyers taking a threshold form: $\sigma^\mu(w) = b^1$ for $w \geq \xi$ and $\sigma^\mu(w) = b^0$ for $w < \xi$ for some threshold $\xi \in (0, 1)$. Such threshold strategies are monotone and deterministic. This restriction to threshold strategies allows us to adopt the approach of finding a communicative equilibrium of the unobserved mechanism design game through solving a constrained optimization problem. Since the characterization of equilibrium communication in Section 4 is partial, in the absence of this restriction, even with two equilibrium messages by uninformed buyers, by Theorem 1 we have a fixed-point problem instead of a constrained optimization problem. After we present the main result (Theorem 2), in Section 5.4 we show that how to construct non-threshold communicative equilibria using equilibria with threshold strategies, and conversely, for any non-threshold communicative equilibrium, there is an equilibrium with a threshold strategy with the same payoffs for all informed and uninformed buyers and the same revenue for the seller.

We refer to communicative equilibria with threshold strategies by uninformed buyers as communicative equilibria with targeted offers, because in equilibrium the seller commits to mechanism that targets take-it-or-leave-it offers to only interested uninformed buyers, i.e., those with valuations above $\xi$ who in equilibrium signal their interests in receiving an offer with message $b^1$. When $\xi = 0$, the equilibrium degenerates into the uncommunicative equilibrium constructed in Li and Peters (2021). For any $\xi > 0$, however, since the commitment to not making an offer to uninformed buyers sending $b^0$ is unobservable to uninformed buyers, the seller faces the temptation to deviate and modify the mechanism by making an offer to uninformed buyers when this is the only profitable option.[5] We show by construction that it is informed buyers with low valuations who discipline the seller and ensure that the seller does not deviate. Of course, this construction relies on the existence of informed buyers, that is, on the assumption of $\alpha < 1$. Indeed, when $\alpha = 1$, the characterization of equilibrium communication given by Lemma 2 remains valid, but in equilibrium the seller makes an offer with probability one and so there is no message $b^0$ that generates no offers. Furthermore, under the assumption that the revenue function $\pi(\cdot)$ is strictly concave, using a similar argument as in Lemma 2 we can easily show that in any equilibrium the lowest message $b^1$ generates the offer $r^*$, which is the unique maximizer of $\pi(\cdot)$, and that the equilibrium revenue is equal to $\pi(r^*)$.[6]

To save notation, we will not index the variables with $\xi$, and we drop $\sigma^\mu$ from all relevant expressions. By Theorem 1, we only need to find an optimal direct mechanism given the threshold strategy $\sigma^\mu$, such that $b^1$ generates a single offer greater than $\xi$ and $b^0$ generates no offers. Based on the insights from Li and Peters (2021), we will construct an optimal equal priority auction, in which informed buyer with valuations on some pooling interval have the same allocation priority as interested uninformed, conditional on no offers being made to uninterested uninformed buyers.[7] This construction is essentially the same as the

[5]Such incentive issue does not exist if uninformed buyers "walk away" after the seller commits to a mechanism. In this alternative model, the optimal equal priority auction characterized by Lemma 4 below is an optimal direct mechanism by the same proof of Theorem 2, without the construction of the multiplier function $\lambda(\cdot; b^0)$. Further, by Proposition 1, in this model the seller's revenue increases with the threshold $\xi$ so long as the optimal equal priority auction is constrained.

[6]By construction, we can establish the existence of an equilibrium with any countable number of offers generated. The detailed proof is available upon request.

[7]It immediately follows from the optimality of the equal priority auction that the offer to interested

characterization of the optimal equal priority auction in Li and Peters (2021), with a strictly positive $\xi$ instead of $\xi = 0$. We then show that for sufficiently small $\alpha$ or $\xi$, the constructed equal priority auction is optimal among all direct mechanisms given $\sigma^{\mu}$; in particular, it is optimal not to make an offer to uninterested uninformed buyers, even when it is the only option. Part of the argument uses the same Lagrangian relaxation approach as in Li and Peters (2021) with respect to the incentive condition for informed buyers. The additional argument has to do with the incentive condition for the seller, to ensure that the information rent needed to prevent informed buyers with low valuations from claiming that they are uninterested uninformed buyers is greater than the revenue gain from making an offer to uninterested buyers.

## 5.1 Optimal equal priority auctions

As in Li and Peters (2021), an equal priority auction consists of a pooling interval $[\underline{w}, \overline{w}]$ of valuations for informed buyers who have the same probability of getting the good as interested uninformed buyers, a take-it-or-leave-it offer $t$ to interested uninformed buyers, and a reserve price $r$ for informed buyers when there are no interested uninformed buyers, satisfying $r \leq \underline{w} \leq \overline{w}$. An equal priority auction $\{r, \underline{w}, \overline{w}; t\}$ is a direct mechanism that does not make an offer an uninterested uninformed buyer, even when doing so is the only option for the seller. Otherwise, the allocation and offer rules are the same as in Li and Peters (2021). It suffices here to specify the trading probabilities for informed buyers and interested uninformed buyers, with the transfer rule for informed buyers determined through standard arguments of incentive compatibility with respect to valuations.

Let $m$ be the number of uninformed buyers, including both interested and uninterested uninformed buyers. Among them, let $m^1$ be the number of interested uninformed buyers. The probability of trade function $Q^{\epsilon}(\cdot)$ for an informed buyer is calculated as follows. For $w < r$,

$$Q^{\epsilon}(w) = 0.$$

uninformed buyers is above the threshold $\xi$.

For $w \in [r, \underline{w})$, we have

$$Q^{\epsilon}(w) = \begin{pmatrix} n-1 \\ m \end{pmatrix} ((1-\alpha)F(w))^{n-1-m}(\alpha F(\xi))^m = ((1-\alpha)F(w) + \alpha F(\xi))^{n-1}.$$

For $w > \overline{w}$,

$$Q^{\epsilon}(w) = \begin{pmatrix} n-1 \\ m \end{pmatrix} ((1-\alpha)F(w))^{n-1-m}\alpha^m = ((1-\alpha)F(w) + \alpha)^{n-1}.$$

Finally, for $w \in [\underline{w}, \overline{w}]$,

$$Q^{\epsilon}(w) = \sum_{m=0}^{n-1} B(m; n-1, \alpha) \sum_{k=0}^{n-1-m} B_k^{n-1-m}(\underline{w}, \overline{w}) \sum_{m^1=0}^{m} B_{m_1}^{m} \frac{1}{m^1 + k + 1},$$

where

$$B_k^{n-1-m}(\underline{w}, \overline{w}) = \begin{pmatrix} n-1-m \\ k \end{pmatrix} (F(\overline{w}) - F(\underline{w}))^k F^{n-1-m-k}(\underline{w})$$

is the probability that $k$ out of $n-1-m$ informed buyers have valuations in the interval $[\underline{w}, \overline{w}]$, and

$$B_{m^1}^{m} = \begin{pmatrix} m \\ m^1 \end{pmatrix} (1 - F(\xi))^{m^1} F^{m-m^1}(\xi)$$

is the probability that $m^1$ out of $m$ uninformed buyers send message $b^1$. By construction, $Q^{\epsilon}(w)$ is constant for all $w \in [\underline{w}, \overline{w}]$. The following lemma provides a convenient formula; the proof is straightforward and omitted.

**Lemma 3** *The allocation probability $Q^{\epsilon}(w)$ under an equal priority auction $\{r, \underline{w}, \overline{w}; t\}$ is*

$$Q^{\epsilon}(w) = \frac{((1-\alpha)F(\overline{w}) + \alpha)^n - ((1-\alpha)F(\underline{w}) + \alpha F(\xi))^n}{n((1-\alpha)(F(\overline{w}) - F(\underline{w})) + \alpha(1 - F(\xi)))}.$$

*for all $w \in [\underline{w}, \overline{w}]$.*

Given the above allocation function $Q^\epsilon(w)$, the payoff for an informed buyer is

$$U^\epsilon(w) = \int_0^w Q^\epsilon(x)dx,$$

and the revenue from informed buyers is

$$n(1-\alpha)\int_r^1 Q^\epsilon(w)\phi(w)f(w)dw.$$

By construction, the probability an interested uninformed buyer generates the offer $t$ with message $b^1$ is the same as the probability of trade $Q^\epsilon(w)$ for an informed buyer with any valuation $w \in [\underline{w}, \overline{w}]$. Since there is a single offer $t$, we will use the notation $\chi$ for the probability of message $b^1$ generating $t$. The payoff for an interested uninformed buyer is

$$U^\mu(w; b^1) = \chi \max\{w - t, 0\}.$$

So long as $t \geq \xi$, by (7), the seller's revenue from uninformed buyers is

$$\left(\sum_{m=1}^{n} B(m; n, \alpha) \sum_{k=0}^{n-m} B_k^{n-m}(\underline{w}, \overline{w}) \sum_{m^1=1}^{m} B_{m_1}^m \frac{m^1}{m^1 + k + 1}\right) \frac{\pi(t)}{1 - F(\xi)}.$$

Using the same arguments as in Lemma 3, we can show that the above reduces to

$$n\alpha\chi\pi(t).$$

The above does not directly depend on $\xi$. Although a greater $\xi$ increases the probability of an interested uninformed buyer accepts a given offer $t$ (provided that $t > \xi$), it decreases the probability that a randomly selected uninformed buyer is interested. These two effects cancel out, and the seller's revenue from uninformed buyers depends on $\xi$ only through $\chi$. For $t < \xi$, the revenue from from uninformed buyers is simply $n\alpha\chi t$.

The incentive condition for an informed buyer not to pretend to be an interested uninformed buyer is

$$\int_0^w Q^\epsilon(x)dx \geq \chi \max\{w - t, 0\}$$

for all $w \in [0,1]$. Since by construction $U^{\epsilon}(w)$ is strictly convex for $w \in [r, \underline{w}]$ and for $w \geq \overline{w}$ and $U^{\mu}(w; b^1)$ is piece-wise linear, the above is satisfied for all $w \in [0,1]$ if and only if it is satisfied at $w = \underline{w}$:

$$\int_r^{\underline{w}} Q^{\epsilon}(x)dx \geq \chi \max\{\underline{w} - t, 0\} \tag{11}$$

The following lemma characterizes the optimal equal priority auction $\{r, \underline{w}, \overline{w}; t\}$ with $r \leq \underline{w} \leq \overline{w}$ that maximizes the sum of the seller's revenue from informed and uninformed buyers subject to the incentive compatibility constraint (11). We refer to it as $\xi$-optimal equal priority auction. The proof is straightforward and omitted. We assume that the revenue function $\pi$ is strictly concave, with $r^* \in (0,1)$ the unique maximizer.

**Lemma 4** *Suppose that $\pi(\cdot)$ is strictly concave, and $\sigma^{\mu}$ is a strategy for uninformed buyers with threshold $\xi \leq r^*$. If an equal priority auction $\{r, \underline{w}, \overline{w}; t\}$ is $\xi$-optimal, then (11) binds, $0 < r < r^* < t < \underline{w} < \overline{w} < 1$, and*

$$\alpha(\pi(t) - (1 - F(\xi))\phi(\overline{w})) =$$
$$(1-\alpha)\Big((\underline{w} - t)(\phi(\overline{w}) - \phi(\underline{w}))f(\underline{w}) + \int_{\underline{w}}^{\overline{w}} f(w)(\phi(\overline{w}) - \phi(w))dw\Big); \tag{12}$$

$$\phi(r)f(r) + (\phi(\overline{w}) - \phi(\underline{w}))f(\underline{w}) = 0; \tag{13}$$

$$\alpha\pi'(t) + (1-\alpha)(\phi(\overline{w}) - \phi(\underline{w}))f(\underline{w}) = 0. \tag{14}$$

Conditions (12), (13) and (14), together with binding (11), are first order necessary conditions for a constrained equal priority auction $\{r, \underline{w}, \overline{w}; t\}$ to be optimal; our Lagrangian relaxation approach to constructing an optimal direct mechanism is based on these conditions. The assumption of $\xi \leq r^*$ is sufficient to imply an optimal equal priority auction binds the incentive condition (11).[8] This is because when $\xi \leq r^*$, the conditional revenue for any

[8]It is also necessary for Theorem 2. When or not the optimal equal priority auction $\{r, \underline{w}, \overline{w}; t\}$ is constrained or unconstrained, if $\xi > r^*$, the seller could make an offer $p^{\mu}(b^0)$ between $r^*$ and $\xi$ to uninterested buyers when all buyers send message $b^0$. This deviation from $\{r, \underline{w}, \overline{w}; t\}$ is profitable for the seller. It will not attract informed buyers, if the offer is made with a small enough probability to a randomly selected uninterested buyer.

$p \geq \xi$ from an offer to an interested uninformed buyer, given by

$$\pi(p|b^1) = \frac{\pi(p)}{1 - F(\xi)},$$

achieves the maximum at $p = r^*$. Since $r^*$ is the optimal reserve price for informed buyers if the incentive condition (11) is slack, an unconstrained optimal equal priority auction would violate the incentive condition (11).[9]

## 5.2 Optimal mechanisms

The main result in this section is that for $\alpha$ or $\xi$ sufficiently small, the optimal equal priority auction $\{r, \underline{w}, \overline{w}; t\}$, which is conditional on not making an offer to uninterested uninformed buyers, characterized by Lemma 4, is in fact an optimal direct mechanism. In particular, for $\alpha$ or $\xi$ sufficiently small, it is optimal for the seller not to make an offer to uninterested buyers, even when it is the only profitable option. The reason for this is that, when $\alpha$ or $\xi$ is small, any revenue gain from making such an offer is outweighed by the incentive cost of preventing informed buyers with low valuations from claiming to be uninterested buyers. The latter is represented by the negative virtual valuation $\phi(w)$ of informed buyers with any valuation $w \leq r^*$.

We establish the main result using the Lagrangian relaxation approach that captures the above intuition. The rough idea is as follows. We construct two non-negatively valued multiplier functions, one for the continuum of incentive compatibility constraints that for each valuation $w \in [0, 1]$ an informed buyer's expected payoff $U^\epsilon(w)$ from truthfully reporting $w$ is greater than or equal to the deviation payoff $U^\mu(w; b^1)$ from pretending to be an interested uninformed buyer, and another for the continuum of incentive compatibility constraints that $U^\epsilon(w)$ is greater than or equal to the deviation payoff $U^\mu(w; b^0)$ from pre-

[9]An unconstrained optimal equal priority auction is given by $r = r^*$, $\underline{w} = \overline{w} = \hat{w}$, and $t = \max\{\xi, r^*\}$, such that $(1 - F(\xi))\phi(\hat{w}) = \pi(t)$. This is the optimal equal priority auction if it satisfies incentive condition (11). Otherwise, the optimal equal priority auction is constrained. For $\xi \leq r^*$, it is given by Lemma 4. For $\xi > r^*$ the optimal equal priority auction can still be constrained, in which case the characterization of Lemma 4 holds except that the first order condition (14) with respect to $t$ is replaced by

$$\alpha\pi'(t) + (1 - \alpha)(\phi(\overline{w}) - \phi(\underline{w}))f(\underline{w}) \leq 0,$$

and $t \geq \xi$, with complementary slackness.

tending to be an uninterested uninformed buyer. These two multiplier functions, denoted as $\lambda(w; b^1)$ and $\lambda(w; b^0)$ respectively, are chosen to ensure that the direct mechanism constructed from the optimal equal priority auction $\{r, \underline{w}, \overline{w}; t\}$ characterized by Lemma 4 is a point-wise maximizer of the Lagrangian.

In particular, as in Li and Peters (2021), for each $w \in [\underline{w}, \overline{w}]$, a properly weighted sum of the incentive benefit $\lambda(w; b^1)$ and the revenue $\phi(w)$ from increasing the allocation for an informed buyer with valuation $w$ is equal to a weighted sum of the total incentive cost $\int_0^1 \lambda(x; b^1)dx$ and the revenue $\pi(t)$ from increasing the allocation for interested uninformed buyers. As a result, the Lagrangian is maximized by giving an equal priority to informed buyers with valuations on $[\underline{w}, \overline{w}]$ and interested uninformed buyers. At the same time, for each $w \leq r$, a weighted sum of the incentive benefit $\lambda(w; b^0)$ and the revenue $\phi(w)$ from increasing the allocation for an informed buyer with valuation $w$, and a weighted sum of the incentive cost $\int_0^1 \lambda(x; b^0)dx$ and the revenue from increasing the allocation for uninterested uninformed buyers, are both non-positive. This implies that the Lagrangian is maximized by not giving an offer to informed buyers with valuations below $r$ or uninterested uninformed buyers. Under the optimal equal priority auction $\{r, \underline{w}, \overline{w}; t\}$ characterized by Lemma 4, informed buyers with valuations on $[\underline{w}, \overline{w}]$ are indifferent between truthfully revealing their valuation and pretending to be interested uninformed buyers, while those with valuations smaller than $r$ are also indifferent between truthfully revealing their valuation and pretending to be uninterested uninformed buyers as they receive a payoff of zero either way. By complementary slackness, the direct mechanism constructed from the optimal equal priority auction $\{r, \underline{w}, \overline{w}; t\}$ characterized by Lemma 4 is a point-wise maximizer of the Lagrangian. Since the value of the Lagrangian is an upper-bound of the value of the objective function, which is the sum of (6) and (7), and since the former is achieved by the equal priority auction $\{r, \underline{w}, \overline{w}; t\}$ characterized by Lemma 4, the latter is indeed an optimal direct mechanism given the threshold strategy $\mu$. It then follows from Theorem 1 that the outcome of $\{r, \underline{w}, \overline{w}; t\}$ corresponds to a symmetric perfect Bayesian equilibrium.

Before we define the Lagrangian, we use the concavity assumption on the monopoly revenue function $\pi(w)$ to simplify the analysis for optimal direct mechanism.

$$\left\{ (q^{\epsilon}_m(v^{\epsilon}; b^{\mu}), p^{\epsilon}_m(v^{\epsilon}; b^{\mu}))^{n-1}_{m=0}, (q^{\mu}_m(v^{\epsilon}; b^{\mu}), p^{\mu}_m(v^{\epsilon}; b^{\mu}))^{n}_{m=1} \right\}.$$

We show that, given the threshold form of $\sigma^{\mu}$, concavity of $\pi(w)$ implies that $p^{\mu}_m(v^{\epsilon}; b^{\mu})$ is independent of the profile of valuations $v^{\epsilon}$ of informed buyers, and depends on the profile of messages $b^{\mu}$ only through the message $b_n$ to whom the offer $p^{\mu}_m(v^{\epsilon}; b^{\mu})$ is made. The proof of the lemma below is a generalization of Lemma 2, and is omitted.

**Lemma 5** *If $\pi(\cdot)$ is strictly concave, and uninformed buyers use threshold strategy, then in any optimal direct mechanism, $p^{\mu}_m(v^{\epsilon}; b^{\mu})$ depends only on $b_n$.*

Given the threshold strategy $\sigma^{\mu}$, the only relevant information about the profile of messages from uninformed is the number $m^1$ of buyers that send $b^1$. For notational brevity, we drop the dependence on $\sigma^{\mu}$. By Lemma 5, we can further rewrite direct mechanisms under consideration as

$$\left\{ \left( \left( q^{\epsilon}_{m,m^1}(v^{\epsilon}), p^{\epsilon}_{m,m^1}(v^{\epsilon}) \right)^{m}_{m^1=0} \right)^{n-1}_{m=0}, \left( \left( \left( q^{\mu}_{m,m^1}(v^{\epsilon}; b^j) \right)^{m}_{m^1=0} \right)^{n}_{m=1}, p^{\mu}(b^j) \right)_{j=0,1} \right\},$$

where $p^{\epsilon}_{m,m^1}(v^{\epsilon})$ and $q^{\epsilon}_{m,m^1}(v^{\epsilon})$ are, respectively, offer and offer probability for an informed buyer with valuation $v_1$ in the profile $v^{\epsilon}$ when $m^1$ out of $m$ uninformed buyers are interested, and $q^{\mu}_{m,m^1}(v^{\epsilon}; b^j)$ are offer probability for the uninformed buyer $n$ who sends message $b_n = b^j$, $j = 0, 1$, when the profile of valuations of informed buyers is $v^{\epsilon}$ and, including bidder $n$, $m^1$ out of $m$ uninformed buyers are interested, and $p^{\mu}(b^j)$ is the offer for uninformed buyer $n$ who sends message $b_n = b^j$. The feasibility constraint (1) becomes

$$\sum_{i=1}^{n-m} q^{\epsilon}_{m,m^1}\left(\rho^{\epsilon}_i\left(v^{\epsilon}\right)\right) + m^1 q^{\mu}_{m,m^1}(v^{\epsilon}; b^1) + (m - m^1) q^{\mu}_{m,m^1}(v^{\epsilon}; b^0) \leq 1.$$

For each $m = 0, \ldots, n-1$, define

$$Q^{\epsilon}_m(w) = \mathbb{E}_{v^{\epsilon}_{-1}} \left[ \sum_{m^1=0}^{m} B^m_{m^1} q^{\epsilon}_{m,m^1}(w, v^{\epsilon}_{-1}) \right],$$
$$Q^{\mu}_{m+1}(b^j) = \mathbb{E}_{v^{\epsilon}} \left[ \sum_{m^1=0}^{m} B^m_{m^1} q^{\mu}_{m+1,m^1+j}(v^{\epsilon}; b^j) \right]$$

for each message $b^j$, $j = 0, 1$. We can now write the expected payoff (2) for an informed buyer as

$$U^{\epsilon}(w) = \int_0^w \sum_{m=0}^{n-1} B(m; n-1, \alpha) Q^{\epsilon}_m(x)\, dx,$$

and the seller's revenue (6) from all informed buyers as

$$n(1-\alpha) \int_0^1 \sum_{m=0}^{n-1} B(m; n-1, \alpha) Q^{\epsilon}_m(w) \phi(w) f(w) dw.$$

From (3), the expected payoff for an interested uninformed buyer is

$$U^{\mu}(w; b^1) = \sum_{m=0}^{n-1} B(m; n-1, \alpha) Q^{\mu}_{m+1}(b^1) \max\left\{ w - p^{\mu}(b^1), 0 \right\},$$

and the expected payoff for an uninterested uninformed buyer is

$$U^{\mu}(w; b^0) = \sum_{m=0}^{n-1} B(m; n-1, \alpha) Q^{\mu}_{m+1}(b^0) \max\left\{ w - p^{\mu}(b^0), 0 \right\}.$$

From (7), the seller's revenue from all interested uninformed buyers is

$$n\alpha \sum_{m=0}^{n-1} B(m; n-1, \alpha) Q^{\epsilon}_{m+1}(b^1) p^{\mu}(b^1) \frac{1 - F(p^{\mu}(b^1))}{1 - F(\xi)},$$

while the seller's revenue from all uninterested uninformed buyers is

$$n\alpha \sum_{m=0}^{n-1} B(m; n-1, \alpha) Q^{\epsilon}_{m+1}(b^0) p^{\mu}(b^0) \frac{F(\xi) - F(p^{\mu}(b^1))}{F(\xi)}.$$

Using $\lambda(w; b^j)$ for the incentive compatibility constraints

$$U^\epsilon(w) \geq U^\mu(w; b^j)$$

for each message $b^j$, $j = 0, 1$, we can now write the Lagrangian as

$$\begin{aligned}
& n(1-\alpha) \sum_{m=0}^{n-1} B(m; n-1, \alpha) \int_0^1 Q_m^\epsilon(w)\phi(w)f(w)dw \\
& + n\alpha \sum_{m=0}^{n-1} B(m; n-1, \alpha) \left( Q_{m+1}^\mu(b^1)\pi(p^\mu(b^1)) + Q_{m+1}^\mu(b^0)p^\mu(b^0)(F(\xi) - F(p^\mu(b^0))) \right) \\
& + \sum_{j=0,1} \int_0^1 \lambda(w; b^j) \sum_{m=0}^{n-1} B(m; n-1, \alpha) \left( \int_0^w Q_m^\epsilon(x)dx - Q_{m+1}^\mu(b^j) \max\left\{w - p^\mu(b^j), 0\right\} \right) dw.
\end{aligned}$$

Using integration by parts, we can further rewrite the above Lagrangian as

$$\sum_{m=0}^{n-1} B(m; n-1, \alpha) \left( \int_0^1 K^\epsilon(w) Q_m^\epsilon(w) f(w) dw + K^\mu(b^1) Q_{m+1}^\mu(b^1) + K^\mu(b^0) Q_{m+1}^\mu(b^0) \right), \quad (15)$$

where

$$K^\epsilon(w)f(w) = n(1-\alpha)\phi(w)f(w) + \int_w^1 (\lambda(x; b^1) + \lambda(x; b^0))dx,$$

for each $w \in [0, 1]$, and

$$\begin{aligned}
K^\mu(b^1) &= n\alpha\pi(p^\mu(b^1)) - \int_0^1 \lambda(w; b^1) \max\{w - p^\mu(b^1), 0\}dw, \\
K^\mu(b^0) &= n\alpha p^\mu(b^0)(F(\xi) - F(p^\mu(b^0))) - \int_0^1 \lambda(w; b^0) \max\{w - p^\mu(b^0), 0\}dw.
\end{aligned}$$

The expressions of $K^\epsilon(w)$, $K^\mu(b^1)$ and $K^\mu(b^0)$ have straightforward interpretations. For informed buyers, $K^\epsilon(w)$ represents the total contribution to the Lagrangian when allocation to an informed buyer with valuation $w$ is increased: the first term in $K^\epsilon(w)$ is the virtual surplus gained by the seller, while the second term is the incentive benefit, because increasing the allocation to informed buyer with valuation $w$ raises the expected payoff to informed buyers with all greater valuations and relaxes the incentive constraints that informed buyers weakly prefer truthfully revealing their valuation to pretending to be interested or uninterested uninformed buyers. For interested uninformed buyers, $K^{\mu}(b^1)$ represents the total contribution to the Lagrangian when the allocation to an interested uninformed buyer is increased: the first term is the revenue gained by the seller, while the second term is the incentive cost, because increasing the allocation to interested buyers increases the deviation payoff informed buyers obtain when they pretend to be interested uninformed buyer. Similarly, $K^{\mu}(b^0)$ represents the total contribution to the Lagrangian when the allocation to an uninterested uninformed buyer is increased, with the first term being the revenue for the seller, and the second term the incentive cost.

**Theorem 2** *Suppose that $\pi(\cdot)$ is strictly concave, $\sigma^{\mu}$ is a threshold strategy for uninformed buyers with threshold $\xi$, and $\{r, \underline{w}, \overline{w}; t\}$ is a $\xi$-optimal equal priority auction. For all $\xi$ such that*

$$(1-\alpha)(F(r^*) - F(\xi)) \geq \xi f(\xi), \tag{16}$$

*$\{r, \underline{w}, \overline{w}; t\}$ is an optimal direct mechanism.*

We outline the proof here and leave the details to the appendix. Suppose that $\{r, \underline{w}, \overline{w}; t\}$ is a $\xi$-optimal equal priority auction. Suppose also that $\xi$ satisfies (16); this implies that $\xi < r^*$. Then, by Lemma 4, the critical incentive compatibility constraint (11) binds and the first order conditions (12), (13), and (14) are satisfied. We will show below that for two appropriately chosen multiplier functions, the allocations given by $\{r, \underline{w}, \overline{w}; t\}$, together with $t$ to interested uninformed buyers and with no offers to uninterested uninformed buyers, are a point-wise maximizer of the Lagrangian (15), among all feasible, weakly increasing allocations for informed buyers and all offers to uninformed buyers.

The construction of the multiplier function $\lambda(w; b^1)$ is essentially the same as in Li and Peters (2021). The function takes positive values only for valuations $w$ on the pooling interval $[\underline{w}, \overline{w}]$, where the incentive condition for informed buyers not to pretend to be interested uninformed buyers is binding under the allocation rule of the $\xi$-optimal equal priority auction. For $w \in [\underline{w}, \overline{w}]$, the value of $\lambda(w; b^1)$ is constructed such that $K^{\epsilon}(w)$ takes the same constant value as the maximal value of $K^{\mu}(b^1)$, achieved when $p^{\mu}(b^1)$ is set to $t$. We show this construction also ensures that the resulting $K^{\epsilon}(w)$ is greater than the maximal value of $K^{\mu}(b^1)$ for $w > \overline{w}$ and is smaller for $w \in (r, \underline{w})$. Then, the allocation rule under the $\xi$-

optimal equal priority auction is a point-wise maximizer of the Lagrangian (15) for assigning the good among informed buyers with valuations greater than $r$ and interested uninformed buyers.

The multiplier function $\lambda(w; b^0)$ is constructed to satisfy the seller's incentive condition that, given the strategy of uninterested uninformed buyers with threshold $\xi$, it is optimal for the seller not to make an offer to them. This is because of the great incentive cost of making an offer $p^\mu(b^0)$ for violating the condition that informed buyers do not pretend to be uninterested, represented by the second term in $K^\mu(b^0)$. More precisely, $\lambda(w; b^0)$ takes positive value only for valuations below $r$, where the incentive condition for informed buyers not to pretend to be uninterested uninformed buyers is binding under the allocation rule of the $\xi$-optimal equal priority auction, because informed buyers with valuations below $r$ receive the same payoff of 0 by reporting their valuation truthfully and by pretending to be uninterested. For $w \leq r$, the value of $\lambda(w; b^0)$ is constructed such that $K^\epsilon(w)$ takes on the constant value of 0, using the first order conditions in Lemma 4. When (16) holds, we show that the maximum of $K^\mu(b^0)$ over $p^\mu(b^0)$ is non-positive.[10] As a result, the allocation rule under the $\xi$-optimal equal priority auction with respect to not making an offer to informed buyers with valuation below $r$ or uninterested uninformed buyers is a point-wise maximizer of the Lagrangian (15).

Since transfers to informed buyers can be constructed to make truthful reporting of valuations incentive compatible given that informed buyers have weakly increasing allocations, we have a direct mechanism from $\{r, \underline{w}, \overline{w}; t\}$, with no offers to uninterested uninformed buyers, that maximizes the Lagrangian (15). Further, by construction, for each message $j = 0, 1$ by uninformed buyers, the product of $\lambda(w; b^j)$ and $U^\epsilon(w) - U^\mu(w; b^j)$ is 0 for all $w$. This implies that the maximum value of the Lagrangian (15), which is an upper bound of the seller's revenue among all feasible, incentive compatible mechanisms, is achievable through the direct mechanism constructed from $\{r, \underline{w}, \overline{w}; t\}$. It follows that the direct mechanism is optimal. By Theorem 1, it corresponds to a communicative equilibrium.

[10] Although condition (16) is sufficient but not necessary, it is stated in terms of the primitives of the model.

## 5.3 Comparative statics

Theorem 2 shows that in any unobserved mechanism design game, there is a continuum of communicative equilibria where uninformed buyers use a threshold strategy, represented by threshold values of $\xi$ that satisfy (16). The intuition behind condition (16) is the following. The equal priority auction $\{r, \underline{w}, \overline{w}; t\}$ characterized in Lemma 4 by construction does not make an offer to uninterested uninformed buyers. In order to make this commitment by the seller credible, the potential revenue that could be made by making an offer to these buyers when they are the only option for the seller must be out-weighed by the incentive cost of enticing informed buyers with valuations below the reserve price $r$ to pretend to be uninterested. This is more unlikely to be true, if the probability $\alpha$ that a buyer is uninformed is small, or if the threshold $\xi$ for expressing interests is low, which is what condition (16) requires.

In this subsection, we use the first order conditions for an optimal equal priority auction in Lemma 4 to show that the seller's revenue is decreasing $\alpha$ and increasing in $\xi$. By Lemma 4, for each $\xi \leq r^*$, an optimal equal priority auction $\{r, t, \underline{w}, \overline{w}\}$ binds (11) and satisfies the first order conditions (12), (13) and (14). The arguments for the proposition below are straightforward applications of the Envelope Theorem, and do not directly depend on whether $\xi$ satisfies (16). They are relegated to the appendix.

**Proposition 1** *Suppose $\pi(\cdot)$ is strictly concave, and for each $\alpha$ and $\xi$ on some intervals there is an equal priority auction $\{r, \underline{w}, \overline{w}; t\}$ that binds (11), and satisfies (12), (13) and (14). Then, the seller's revenue from the auction is decreasing in $\alpha$ and increasing in $\xi$.*

Between the two extremes of $\alpha = 0$ and $\alpha = 1$, the seller clearly prefers the former, with the revenue given by Myerson's optimal auction with $r^*$, to the latter, with the revenue given by making the optimal take-it-or-leave-it offer $r^*$ from a randomly selected buyer. As $\alpha$ increases, in the revenue from an optimal equal priority auction the revenue from interested uninformed buyers replaces the revenue from informed buyers. This implies that the direct effect of an increase in $\alpha$ on the seller's revenue is negative. However, there is an indirect effect through the effect of an increase in $\alpha$ on $\chi$, the probability that the seller makes an offer to a buyer in the equal priority pool. From the proof in the appendix, this effect is ambiguous. What the proof of Proposition 1 reveals is that, despite of this ambiguity, the seller's revenue from an optimal interior equal priority auction is monotonically decreasing in $\alpha$.

An increase in $\xi$ directly increases the seller's revenue from informed buyers with valuations between $r$ and $\underline{w}$, by effectively raising the priority level of those informed buyers. It also indirectly increases the seller's revenue from both informed buyers with valuations on the pooling interval $[\underline{w}, \overline{w}]$ and interested uninformed buyers, by increasing $\chi$, the probability that an offer is made to these buyers with the same priority level. The overall effect is ambiguous on the critical incentive condition that an informed buyer with valuation $\underline{w}$ is weakly worse off by pretending to be an interested uninformed buyer, because an increase in $\xi$ raises the interim payoffs of the informed buyer both from the auction and in deviation. However, we show that the increase in the seller's revenue from uninformed buyers alone more than compensates for the increase in the incentive cost due to the increase in the deviation payoff of the informed buyer with valuation $\underline{w}$.

## 5.4 Discussion

The constructive proof we have come up with for Theorem 2 leaves open the question of whether communicative equilibria with threshold $\xi$ that violates (16) exist. The answer is yes, but only because condition (16) is sufficient but not necessary for a key property required for showing that not making an offer to uninterested uninformed buyers maximizes the relaxed Lagrangian (15) – that the maximum of $K^\mu(b^0)$ over $p^\mu(b^0)$ is non-positive. We have chosen $\lambda(w; b^0)$ to be 0 for all $w > r$ and positive for $w \leq r$ such that $K^\epsilon(w)$ is 0 for all $w \leq r$. This function shows up both in the incentive cost of making an offer to an uninterested buyer, which is the negative second term in $K^\mu(b^0)$, and in the incentive benefit of making an offer to an informed buyer with valuation lower than $r$, which is part of the positive second term in $K^\epsilon(w)$. Since $K^\epsilon(w)$ is required to be non-positive for all $w \leq r$ – recall that $r$ is the reserve price for informed buyers when there are no interested uninformed buyers – our choice of $\lambda(\cdot; b^0)$ is point-wise maximal. For this choice of $\lambda(\cdot; b^0)$, not making an offer to uninterested uninformed buyers maximizes the relaxed Lagrangian (15) only if the maximum of $K^\mu(b^0)$ over $p^\mu(b^0)$ is non-positive. Since the choice of $\lambda(\cdot; b^0)$ is maximal, it is both sufficient, and necessary, for the optimal equal priority auction $\{r, \underline{w}, \overline{w}; t\}$ to be the optimal direct mechanism that the maximum of $K^{\mu}(b^0)$ over $p^{\mu}(b^0)$ is non-positive for our choice of $\lambda(\cdot; b^0)$.

For communicative equilibria of the unobserved mechanism design game with two distinct messages, $b^0$ that generates no offers and $b^1$ that generates a single offer, we have restricted to strategies $\sigma^{\mu}$ of uninformed buyers that take a threshold form. Communicative equilibria with non-threshold strategies also exist. Consider $\sigma^{\mu}(w)$ given by a function $h(w)$, representing the probability that an uninformed buyer with valuation $w$ chooses $b^0$, with $1-h(w)$ the probability of the buyer choosing $b^1$. Unlike in our search for communicative equilibria with threshold strategies, here we deal with the fixed-point problem by conjecturing that communicative equilibria with non-threshold strategies can be supported by optimal equal priority auctions. Specifically, by Theorem 1 (more specifically, Lemma 2), we need $\sigma^{\mu}$ to be a best response to some $\{r, \underline{w}, \overline{w}; t\}$, which means that $h(w) = 0$ for all $w > t$. This is satisfied by any threshold equilibrium so long as the threshold $\xi$ is smaller than $t$; here we proceed to construct a non-threshold equilibrium without imposing any additional restrictions on $h(w)$. Let

$$H = \int_0^1 h(w)f(w)dw.$$

The conditional revenue function for message $b^1$ is then

$$\pi(p|b^1) = \frac{\int_p^1 w(1-h(w))f(w)dw}{1-H}.$$

The formula for $\tilde{\chi}$, the probability that $b^1$ generates the offer $t$ in an equal priority auction $\{r, \underline{w}, \overline{w}; t\}$, is still given by Lemma 3, with $F(\xi)$ replaced by $H$ and $1-F(\xi)$ replaced with $1-H$. With this formula, the revenue from interested uninformed buyer is

$$n\alpha\tilde{\chi}(1-H)\pi(p|b^1).$$

The counterpart of Lemma 4 is: If $\pi(\cdot)$ is strictly concave, and an equal priority auction $\{r, \underline{w}, \overline{w}; t\}$ is interior with $0 < r < r^* < t < \underline{w} < \overline{w} < 1$ and binds (11), then the first order condition (12), (13) and (14) continue to hold, with the only change being that $1-H$ replaces

$1-F(\xi)$ in (12). In particular, $\pi(t)$ in (12) and $\pi'(t)$ in (14) both remain unchanged because $h(w)=0$ for all $w>t$. The construction of $\lambda(w;b^1)$ for the incentive condition that informed buyers do not wish to pretend to be interested uninformed buyers, and the construction of $\lambda(w;b^0)$ for the incentive condition that uninformed buyers do not wish to pretend to be uninterested buyers, are both unchanged from the corresponding constructions in the proof of Theorem 2.[11] The only change is that, in place of condition (16), we need to ensure that $h(\cdot)$ to be such that the maximum of $K^\mu(b^0)$ over $p^\mu(b^0)$ is non-positive. This requires that $h(w)=0$ for all $w>r$; otherwise, the revenue from uninterested uninformed buyers (the first term in $K^\mu(b^0)$) would be positive for any $p^\mu(b^0)$ just above $r$, but the incentive cost of making such an offer (the second term in $K^\mu(b^0)$) would be 0. Given this, $K^\mu(b^0)$ can be written as

$$n\alpha p^\mu(b^0)\int_{p^\mu(b^0)}^r h(w)f(w)dw - n(1-\alpha)\int_{p^\mu(b^0)}^r (\phi(r)f(r)-\phi(w)f(w))dw, \tag{17}$$

and we require the above has a maximum value over $p^\mu(b^0)$ that is non-positive.

Now we can easily use a communicative equilibrium with a threshold strategy by uninformed buyers to construct another communicative equilibrium with the same supporting optimal equal priority auction $\{r,\underline{w},\overline{w};t\}$, but with a non-threshold strategy. For example, start with a communicative equilibrium with threshold $\xi$ that satisfies (16), and define a non-threshold strategy $\sigma^\mu$ with $h(\cdot)$, given by

$$h(v)=\begin{cases} F(\xi)/F(\tilde{\xi}) & \text{if } w\le\tilde{\xi} \\ 0 & \text{if } w>\tilde{\xi} \end{cases}$$

for some $\tilde{\xi}\in(\xi,r)$. By construction, $H=F(\xi)$. The first term in $K^\mu(b^0)$ given by (17) is increasing in $\tilde{\xi}$ for any $p^\mu(b^0)$. So long as $\xi$ satisfies (16) with slack, there are values of $\tilde{\xi}$ such that the maximum value of $K^\mu(b^0)$ over $p^\mu(b^0)$ that is non-positive. This gives us a non-threshold equilibrium with $h$ that has the same supporting optimal equal priority auction $\{r,\underline{w},\overline{w};t\}$ as the threshold equilibrium with $\xi$. Conversely, given a supporting optimal equal

[11] It remains true that $p^\mu(b^1)=t$ maximizes $K^\mu(b^1)$, where $1-H$ replaces $1-F(\xi)$ in the maximized value.

priority auction $\{r, \underline{w}, \overline{w}; t\}$ and a non-threshold equilibrium with $h$ such that $h(w) = 0$ for all $w \geq r$, we can define $\xi$ such that $F(\xi) = H$, and construct a threshold equilibrium with $\xi$ and the same supporting optimal equal priority auction $\{r, \underline{w}, \overline{w}; t\}$. This follows immediately by noting that the first term of $K^\mu(b^0)$ given by (17) cannot increase for any $p^\mu(b^0)$, if we change $h(\cdot)$ to $\tilde{h}(w) = 0$ for all $w \geq \xi$ and $\tilde{h}(w) = 1$ otherwise.

We have restricted to communicative equilibria with just two messages, $b^0$ that generates no offers and $b^1$ that generates a single offer. Our partial characterization offers a road map for constructing equilibria with more messages. By Lemma 2, if there are $l$ equilibrium messages used by uninformed buyers that generate offers from the seller's mechanism in addition to a message $b^0$ that generates no offers, then we can order them as $b^0, b^1, \ldots, b^l$, such that each $b^l$, $l \geq 1$, generates offers $t^1, \ldots, t^l$ with probability $\chi^1, \ldots, \chi^l$. For each $l \geq 1$, uninformed buyers with valuation on each interval $(t^l, t^{l+1}]$ can randomize among all messages $b^l$ and higher. However, since each message $b^l$ generates offers $t^1, \ldots, t^l$, the conditional revenue function $\pi(t|b^l)$ must satisfy

$$\max_{t \in [t^j, t^{j+1}]} \pi(t|b^l) = \pi(t^j|b^l) \tag{18}$$

for each $j = 1, \ldots, l-1$. Otherwise, the seller could replace the offer $t^j$ to $b^l$ with a higher offer on $(t^j, t^{j+1}]$, which would relax the incentive condition (5) of informed buyers and raise the seller's revenue, contradicting Theorem 1. The above additional restriction (18) on how uninformed buyers with valuation on each interval $(t^l, t^{l+1}]$ randomize among messages $b^l$ and higher implies that $\pi(t|b^l)$ is weakly decreasing along the sequence of offers $t^1, \ldots, t^l$ generated by message $b^l$. Just as in a communicative equilibrium with two messages $b^0$ and $b^1$ characterized by Theorem 2, it will be the informed buyers who make it credible for the seller's equilibrium mechanism to respond to message $b^l$ from uninformed buyers with a less profitable offer, say $t^2$, rather than $t^1$. We leave it to future research to address the issue of how to construct communicative equilibria with two or more offers made to uninformed buyers.

# 6 Concluding Remarks

Uninformed buyers in our unobserved mechanism design game are rational: they do not observe the seller's equilibrium mechanism but know what equilibrium is being played. This is the same assumption made in the literature on rational inattention that starts with Sim's (2003) seminal work.[12] Following the spirit of this literature, a natural question is what if uninformed buyers can choose to learn about the seller's mechanism at some cost. Further, such learning could be selective, and uninformed buyers with different valuations may have different incentives to learn about the seller's mechanism. More importantly, unlike the literature of rational inattention, however, the object of selective learning by uninformed buyers is simultaneously the object of design by the seller. We leave these difficult modeling issues to future work.

In the communicative equilibria of the unobserved mechanism design game constructed in this paper, uninformed buyers use a simple way to communicate their valuations to the seller. The idea that in a mechanism design problem participating agents may have exogenously restricted ability to communicate with the principal goes back to at least as far as Green and Laffont (1986). A more recent paper, Glazer and Rubinstein (2012), takes the further step of "endogenizing" restrictions on communication by having the principal design mechanisms that exploit boundedly rational procedures used by agents. Our unobservable mechanism design game may be viewed as an alternative framework that explains the appearance of communication restrictions through the assumption that agents do not observe the fully committed mechanism of the principal. Our uninformed buyers are not boundedly rational; the limitation of their communication strategy is an equilibrium result. If uninformed buyers are exogenously restricted to two messages as in our main result about targeted offers but observe the seller's mechanism, we would have an optimal mechanism problem. The seller can do as least as well as in our equilibrium equal priority auction with targeted offers, and should be able to do strictly better. In future work we plan to characterize the solutions to this class of optimal mechanism design problems and contrast the solutions to communicative equilibria in the unobserved mechanism design game.

[12]See for Mackowiak, Matejka and Wiederholt (2021) for a recent survey.

The seller in our model formally has the full commitment power but may face uninformed buyers who do not observe any commitment made by the seller. With the implicit assumption that the seller makes a one-time offer, unobserved mechanism design ends up as a mixture of full commitment with respect to informed buyers, and no commitment with respect to uninformed buyers. In this sense, our framework is related to the literature of mechanism with limited commitment; see, e.g., Bester and Strausz (2001). While full commitment is well understood in the auction environment since Myerson (1981), "auction with no commitment" has not received much attention because no commitment is typically modeled in the literature as a durable goods problem in a dynamic game with repeated offers (Fudenberg, Levine and Tirole, 1985; Gul, Sonnenschein and Wilson, 1986). With one-time offers, and only uninformed buyers, communicative equilibria can still be constructed. The seller is forced to make an offer with probability one, so equilibria with targeted offers as constructed in this paper are impossible. We leave it to future work to completely characterize equilibria in auction with no commitment and their welfare properties.

# 7 Appendix

## 7.1 Proof of Theorem 2

The proof is organized in the following seven steps.

(i) Constructing the two non-negatively valued multiplier functions.

Let $\lambda(w; b^1) = 0$ for all $w \notin [\underline{w}, \overline{w}]$, and let

$$\begin{aligned}\lambda(w; b^1) =& n(1-\alpha)\frac{d}{dw}((\phi(w) - \phi(\overline{w}))f(w)) \\ =& n(1-\alpha)(2f(w) + (w - \phi(\overline{w}))f'(w))\end{aligned}$$

for all $w \in [\underline{w}, \overline{w}]$. Since by assumption $\pi(\cdot)$ is strictly concave, $\phi(w)f(w)$ is strictly increasing in $w$, and thus $\lambda(w; b^1) > 0$ at any $w \in [\underline{w}, \overline{w}]$ such that $f'(w) \leq 0$. By (12) we have $(1 - F(\xi))\phi(\overline{w}) < \pi(t) < \pi(r^*)$. Since $w \geq \underline{w} > t > r^*$, for any $\xi \leq r^*$ we have $\lambda(w; b^1) > 0$ at any $w \in [\underline{w}, \overline{w}]$ such that $f'(w) > 0$. Thus, $\lambda(w; b^1)$ as constructed is non-negative for any

$w$.

Let $\lambda(w; b^0) = 0$ for all $w > r$, and let

$$\lambda(w; b^0) = n(1-\alpha)\frac{d}{dw}(\phi(w)f(w))$$

for all $w \leq r$. Since $\pi(\cdot)$ is strictly concave, $\lambda(w; b^0)$ is non-negative for all $w$.

(ii) Computing $K^\epsilon(w)$.

By construction

$$\int_w^1 \lambda(x; b^1)dx = \begin{cases} 0 & \text{if} \quad w > \overline{w} \\ n(1-\alpha)(\phi(\overline{w}) - \phi(w))f(w) & \text{if} \quad w \in [\underline{w}, \overline{w}] \\ n(1-\alpha)(\phi(\overline{w}) - \phi(\underline{w}))f(\underline{w}) & \text{if} \quad w < \underline{w}, \end{cases}$$

and

$$\int_w^1 \lambda(x; b^0)dx = \begin{cases} 0 & \text{if} \quad w > r \\ n(1-\alpha)(\phi(r)f(r) - \phi(w)f(w)) & \text{if} \quad w \leq r. \end{cases}$$

Thus

$$K^\epsilon(w) = \begin{cases} n(1-\alpha)\phi(w) & \text{if} \quad w > \overline{w} \\ n(1-\alpha)\phi(\overline{w}) & \text{if} \quad w \in [\underline{w}, \overline{w}] \\ n(1-\alpha)\left(\phi(w) + (\phi(\overline{w}) - \phi(\underline{w}))f(\underline{w})/f(w)\right) & \text{if} \quad w \in (r, \underline{w}) \\ 0 & \text{if} \quad w \leq r, \end{cases}$$

where the last case (for $w \leq r$) follows from (13).

(iii) Showing $K^\epsilon(w)$ is strictly increasing for $w > \overline{w}$ and for $w \in (r, \underline{w})$.

By step (ii), for $w > \overline{w}$, the claim follows if we show that $\phi(w)$ is strictly increasing for all $w > \overline{w}$. By definition,

$$\phi(w) = w - \frac{1-F(w)}{f(w)},$$

and so $\phi'(w) > 0$ for all $w$ such that $f'(w) \geq 0$. By strict concavity of $\pi(w)$, we have

$$(\phi(w)f(w))' = \phi'(w)f(w) + \phi(w)f'(w) > 0.$$

Since $\overline{w} > r^*$, we have $\phi(w) > 0$ for all $w > \overline{w}$. The above inequality implies that $\phi'(w)f(w) > -\phi(w)f'(w)$, and thus $\phi'(w) > 0$ for all $w > \overline{w}$ such that $f'(w) < 0$.

By step (ii), for $w \in (r, \underline{w})$, the claim that $K^\epsilon(w)$ is strictly increasing follows if we can show that

$$\tilde{\phi}(w) \equiv \phi(w) + \frac{(\phi(\overline{w}) - \phi(\underline{w}))f(\underline{w})}{f(w)} = w - \frac{1 - F(w) - (\phi(\overline{w}) - \phi(\underline{w}))f(\underline{w})}{f(w)}$$

is strictly increasing. Since

$$(1 - F(\xi))\phi(\overline{w}) < \pi(t) < \pi(r^*) = r^*(1 - F(r^*)),$$

and since $\underline{w} > r^*$, for any $\xi \leq r^*$,

$$(\phi(\overline{w}) - \phi(\underline{w}))f(\underline{w}) = (\phi(\overline{w}) - \underline{w})f(\underline{w}) + 1 - F(\underline{w}) < 1 - F(\underline{w}) < 1 - F(w)$$

for all $w < \underline{w}$. It follows that $\tilde{\phi}'(w) > 0$ for any $w < \underline{w}$ such that $f'(w) \geq 0$. By strict concavity of $\pi(w)$,

$$(\tilde{\phi}(w)f(w))' = \tilde{\phi}'(w)f(w) + \tilde{\phi}(w)f'(w) > 0.$$

By (13), $\tilde{\phi}(w) > 0$ for all $w \in (r, \underline{w})$. It follows that $\tilde{\phi}'(w) > 0$ for all $w \in (r, \underline{w})$ such that $f'(w) < 0$.

(iv) Showing that $p^\mu(b^1) = t$ maximizes $K^\mu(b^1)$, with the maximum value given by

$$K^\mu(b^1) = n\alpha(1 - F(\xi))\phi(\overline{w}) > 0.$$

For any $p^\mu(b^1) < \underline{w}$, using integration by parts and step (ii), we have

$$\begin{aligned}
&\int_0^1 \lambda(w; b^1) \max\{w - p^\mu(b^1), 0\} dw \\
=& -\int_{\underline{w}}^{\overline{w}} (w - p^\mu(b^1)) \; d\left(\int_w^1 \lambda(x; b^1) dx\right) \\
=& n(1-\alpha)\left((\underline{w} - p^\mu(b^1))(\phi(\overline{w}) - \phi(\underline{w}))f(\underline{w}) + \int_{\underline{w}}^{\overline{w}} (\phi(\overline{w}) - \phi(w))f(w)dw\right).
\end{aligned}$$

By (12), we have

$$K^{\mu}(b^1) = n\alpha(1-F(\xi))\phi(\overline{w}) + n\alpha(\pi(p^{\mu}(b^1)) - \pi(t)) + (p^{\mu}(b^1) - t)n(1-\alpha)(\phi(\overline{w}) - \phi(\underline{w}))f(\underline{w}).$$

The above is strictly concave in $p^{\mu}(b^1)$. By (14), it is maximized at $p^{\mu}(b^1) = t$, which is strictly smaller than $\underline{w}$. Any $p^{\mu}(b^1) \geq \underline{w}$ can only decrease $K^{\mu}(b^1)$, and so $p^{\mu}(b^1) = t$ is a global maximizer of $K^{\mu}(b^1)$. Substituting $p^{\mu}(b^1) = t$ in the above expression for $K^{\mu}(b^1)$ we have expression for the maximum value.

(v) Showing that

$$\frac{K^{\epsilon}(w)}{1-\alpha} \geq \frac{K^{\mu}(b^1)}{\alpha(1-F(\xi))} \tag{19}$$

if and only if $w \geq \underline{w}$, with equality for all $w \in [\underline{w}, \overline{w}]$, where for simplicity we continue to use the notation $K^{\mu}(b^1)$ for the maximum value given in step (iv).

For all $w \in [\underline{w}, \overline{w}]$, from step (ii) and step (iv), (19) holds as equality.

For all $w > \overline{w}$, by step (iii),

$$K^{\epsilon}(w) = n(1-\alpha)\phi(w) > n(1-\alpha)\phi(\overline{w}) = K^{\epsilon}(\overline{w}).$$

Thus, (19) holds as a strict inequality for all $w > \overline{w}$.

For all $w \in (r, \underline{w})$, by step (iii),

$$K^{\epsilon}(w) = n(1-\alpha)\left(w - \frac{1-F(w) - (\phi(\overline{w}) - \phi(\underline{w}))f(\underline{w})}{f(w)}\right) < n(1-\alpha)\phi(\overline{w}) = K^{\epsilon}(\underline{w}).$$

Thus, (19) holds strictly in reverse for all $w \in (r, \underline{w})$.

For all $w \leq r$, (19) holds strictly in reverse because $K^{\epsilon}(w) = 0$ by step (ii).

(vi) Showing that if $\xi$ satisfies (16), then the maximum of $K^{\mu}(b^0)$ over $p^{\mu}(b^0)$ is non-positive.

First, we show that $r > \xi$. In step (iii), we have shown that

$$(\phi(\overline{w}) - \phi(\underline{w}))f(\underline{w}) < 1 - F(\underline{w}).$$

Since $\underline{w} > t > r^*$, from (13) we have

$$\phi(r)f(r) > -(1 - F(r^*)). \tag{20}$$

Since $\xi$ satisfies (16) and thus $\xi < r^*$, we have

$$\phi(\xi)f(\xi) \leq (1-\alpha)(F(r^*) - F(\xi)) - (1 - F(\xi)) < -(1 - F(r^*)).$$

By strict concavity of $\pi(\cdot)$, we have $r > \xi$ .

Next, since the first term in $K^\mu(b^0)$ is 0 for all $p^\mu(b^0) > \xi$, to establish the claim we can restrict to $p^\mu(b^0) \leq \xi$. Since $\xi < r$, we have $p^\mu(b^0) < r$. Then, using integration by parts and step (ii) we have

$$\begin{aligned}\int_0^1 \lambda(w; b^0) \max\{w - p^\mu(b^0), 0\} dw = &- \int_{p^\mu(b^0)}^r (w - p^\mu(b^0)) \; d\left(\int_w^1 \lambda(x; b^0) dx\right) \\ =& n(1-\alpha) \int_{p^\mu(b^0)}^r (\phi(r)f(r) - \phi(w)f(w)) dw.\end{aligned}$$

By strict concavity of $\pi(\cdot)$, the above is strictly convex, and thus $K^\mu(b^0)$ is strictly concave. Since the first term in $K^\mu(b^0)$ is 0 at $p^\mu(b^0) = \xi$, a sufficient condition for the maximum of $K^\mu(b^0)$ over $p^\mu(b^0) \leq \xi$ is non-positive is that the derivative of $K^\mu(b^0)$ with respect to $p^\mu(b^0)$ is non-negative at $p^\mu(b^0) = \xi$. This is equivalent to

$$(1-\alpha)\phi(r)f(r) \geq \xi f(\xi) - (1-\alpha)(1 - F(\xi)),$$

which follows from combining (20) and (16).

(vii) Verifying the direct mechanism given by $\{r, \underline{w}, \overline{w}; t\}$ point-wise maximizes the Lagrangian (15).

By step (vi), to maximize the Lagrangian, we set $q^\mu_{m,m^1}(v^\epsilon; b^0) = 0$ for all $m = 1, \ldots, n$, all $m^1 = 0, \dots, m$, and all $v^\epsilon$. We can then rewrite (15) as

$$
\begin{aligned}
&(1-\alpha)^{n-1} \int_0^1 K^\epsilon(w) Q_0^\epsilon(w) f(w) dw + \alpha^{n-1} K^\mu(b^1) Q_n^\mu(b^1) \\
&\quad + \sum_{m=1}^{n-1} \left( \int_0^1 B(m; n-1, \alpha) K^\epsilon(w) Q_m^\epsilon(w) f(w) dw + B(m-1; n-1, \alpha) K^\mu(b^1) Q_m^\mu(b^1) \right).
\end{aligned}
$$

The first term in the above Lagrangian can be further disaggregated:

$$
(1-\alpha)^{n-1} \int_0^1 K^\epsilon(w) \mathbb{E}_{v_{-1}^\epsilon} \left[ q_{0,0}^\epsilon(w, v_{-1}^\epsilon) \right] f(w) dw = (1-\alpha)^{n-1} \mathbb{E}_{v^\epsilon} \left[ \sum_{i=1}^n \frac{K^\epsilon(v_i)}{n} q_{0,0}^\epsilon(\rho_i^\epsilon(v^\epsilon)) \right].
$$

By step (ii), $K^\epsilon(w)$ is a positive constant for $w \in [\underline{w}, \overline{w}]$, strictly increasing for $w > \overline{w}$ and for $w \in (r, \underline{w})$, and 0 for $w \in [\xi, r]$, and strictly negative for $w < \xi$. Thus, it is feasible and point-wise maximizing to set $q_{0,0}^\epsilon(\rho_i^\epsilon(v^\epsilon)) = 1$ if $v_i = \max\{v_1, \dots, v_n\}$ and $v_i > \overline{w}$, or if $v_i = \max\{v_1, \dots, v_n\}$ and $v_i \in [r, \underline{w})$; set $q_{0,0}^\epsilon(\rho_i^\epsilon(v^\epsilon)) = 1/k$ if $v_i \in [\underline{w}, \overline{w}]$, $\max\{v_1, \dots, v_n\} \in [\underline{w}, \overline{w}]$ and $\#\{j : v_j \in [\underline{w}, \overline{w}]\} = k$; and set $q_{0,0}^\epsilon(\rho_i^\epsilon(v^\epsilon)) = 0$ otherwise. This coincides with the allocations under $\{r, \underline{w}, \overline{w}; t\}$ for the case of $m = 0$.

For the second term in the Lagrangian, recall from step (iii) that $K^\mu(b^1) > 0$. It is feasible and point-wise maximizing to set $q_{m^1,n}^\mu(b^1) = 1/m^1$ for all $m^1 = 1, \dots, n$. This coincides with the allocations under $\{r, \underline{w}, \overline{w}; t\}$ for the case of $m = n$.

For each $m = 1, \dots, n-1$ in the third term in the Lagrangian, we disaggregate it as follows:

$$
\begin{aligned}
&\int_0^1 B(m; n-1, \alpha) K^\epsilon(w) \mathbb{E}_{v_{-1}^\epsilon} \left[ \sum_{m^1=0}^m B_{m^1}^m q_{m,m^1}^\epsilon(w, v_{-1}^\epsilon) \right] f(w) dw \\
&\quad + B(m-1; n-1, \alpha) K^\mu(b^1) \mathbb{E}_{v^\epsilon} \left[ \sum_{m^1=0}^{m-1} B_{m^1}^{m-1} q_{m,m^1+1}^\mu(v^\epsilon; b^1) \right] \\
=& B(m; n-1, \alpha) \mathbb{E}_{v^\epsilon} \left[ \sum_{i=1}^{n-m} \frac{K^\epsilon(v_i)}{n-m} \sum_{m^1=0}^m B_{m^1}^m q_{m,m^1}^\epsilon(\rho_i^\epsilon(v^\epsilon)) \right] \\
&\quad + B(m-1; n-1, \alpha) \mathbb{E}_{v^\epsilon} \left[ K^\mu(b^1) \sum_{m^1=1}^m B_{m^1-1}^{m-1} q_{m,m^1}^\mu(v^\epsilon; b^1) \right]
\end{aligned}
$$

When $m^1 = 0$, it is feasible and point-wise maximizing to set $q_{0,0}^\epsilon(\rho_i^\epsilon(v^\epsilon)) = 1$ if $v_i =$

$\max\{v_1,\ldots,v_{n-m}\}$ and $v_i > \overline{w}$, or if $v_i = \max\{v_1,\ldots,v_{n-m}\}$ and $v_i \in [r,\underline{w})$; set $q^\epsilon_{0,0}(\rho^\epsilon_i(v^\epsilon)) = 1/k$ if $v_i \in [\underline{w},\overline{w}]$, $\max\{v_1,\ldots,v_{n-m}\} \in [\underline{w},\overline{w}]$ and $\#\{j : v_j \in [\underline{w},\overline{w}]\} = k$; and set $q^\epsilon_{0,0}(\rho^\epsilon_i(v^\epsilon)) = 0$ otherwise. For each $m^1 = 1,\ldots,n-1$, from step (iv) we have

$$B(m;n-1,\alpha)\frac{K^\epsilon(v_i)}{n-m}B^m_{m^1} \geq B(m-1;n-1,\alpha)\frac{K^\mu(b^1)}{m^1}B^{m-1}_{m^1-1}$$

if and only if $v_i \geq \underline{w}$, with equality for all $w \in [\underline{w},\overline{w}]$. Thus, it is feasible and point-wise maximizing to set $q^\epsilon_{m,m^1}(\rho^\epsilon_i(v^\epsilon)) = 1$ for all $m^1$, if $v_i = \max\{v_1,\ldots,v_{n-m}\}$ and $v_i > \overline{w}$; set $q^\epsilon_{m,m^1}(\rho^\epsilon_i(v^\epsilon)) = q^\mu_{m,m^1}(b^1) = 1/(m^1+k)$ if $v_i \in [\underline{w},\overline{w}]$, $\max\{v_1,\ldots,v_{n-m}\} \in [\underline{w},\overline{w}]$ and $\#\{j : v_j \in [\underline{w},\overline{w}]\} = k$; and otherwise set $q^\epsilon_{m,m^1}(\rho^\epsilon_i(v^\epsilon)) = 0$ and $q^\mu_{m,m^1}(b^1) = 1/(m^1+k)$, where $k = \#\{j : v_j \in [\underline{w},\overline{w}]\}$. This coincides with the allocations under $\{r,\underline{w},\overline{w};t\}$ for the case of $m = 1,\ldots,n-1$.

## 7.2 Proof of Proposition 1

From first order conditions (12), (13) and (14), the implied value of the multiplier for (11) is given by

$$\Lambda = n(1-\alpha)(\phi(\overline{w}) - \phi(\underline{w}))f(\underline{w}) = -n(1-\alpha)\phi(r)f(r) = -n\alpha\pi'(t). \tag{21}$$

The Lagrangian is given by

$$\begin{aligned} n(1-\alpha)\int_r^1 Q^\epsilon(w)\phi(w)f(w)dw + n\alpha\chi\pi(t) \\ + \Lambda\left(\int_r^{\underline{w}}((1-\alpha)F(w)+\alpha F(\xi))^{n-1}dw - \chi(\underline{w}-t)\right). \end{aligned} \tag{22}$$

For comparative statics with respect to $\alpha$, we take partial derivatives of the Lagrangian

(22) with respect to $\alpha$ to get

$$
\begin{aligned}
&\frac{\partial\chi}{\partial\alpha}\left(n(1-\alpha)\int_{\underline{w}}^{\overline{w}}\phi(w)f(w)dw+n\alpha\pi(t)-\Lambda(\underline{w}-t)\right)\\
&+\int_{\overline{w}}^{1}(n-1)(1-F(w))((1-\alpha)F(w)+\alpha)^{n-2}n(1-\alpha)\phi(w)f(w)dw\\
&-\int_{r}^{\underline{w}}(n-1)(F(w)-F(\xi))((1-\alpha)F(w)+\alpha F(\xi))^{n-2}(n(1-\alpha)\phi(w)f(w)+\Lambda)dw\\
&-n\left(\int_{r}^{\underline{w}}((1-\alpha)F(w)+\alpha F(\xi))^{n-1}\phi(w)f(w)dw+\chi\left(\int_{\underline{w}}^{\overline{w}}\phi(w)f(w)dw-\pi(t)\right)\right.\\
&\qquad\left.+\int_{\overline{w}}^{1}((1-\alpha)F(w)+\alpha)^{n-1}\phi(w)f(w)dw\right).
\end{aligned}
$$

Using the formula in Lemma 3 and taking derivatives, we have

$$
\begin{aligned}
\frac{\partial\chi}{\partial\alpha}=&\frac{(1-F(\overline{w}))((1-\alpha)F(\overline{w})+\alpha)^{n-1}+((F(\underline{w})-F(\xi))((1-\alpha)F(\underline{w})+\alpha F(\xi))^{n-1}}{(1-\alpha)(F(\overline{w})-F(\underline{w}))+\alpha(1-F(\xi))}\\
&-\frac{\chi(1-F(\overline{w})+F(\underline{w})-F(\xi))}{(1-\alpha)(F(\overline{w})-F(\underline{w}))+\alpha(1-F(\xi))}.
\end{aligned}
$$

Using (21) and (12), we can rewrite the first term in the derivative of the Lagrangian as

$$
\begin{aligned}
&\frac{\partial\chi}{\partial\alpha}((1-\alpha)(F(\overline{w})-F(\underline{w}))+\alpha(1-F(\xi)))n\phi(\overline{w})\\
=&n\phi(\overline{w})\left((1-F(\overline{w}))((1-\alpha)F(\overline{w})+\alpha)^{n-1}+(F(\underline{w})-F(\xi))((1-\alpha)F(\underline{w})+\alpha F(\xi))^{n-1}\right.\\
&\qquad\left.-\chi(1-F(\overline{w})+F(\underline{w})-F(\xi))\right).
\end{aligned}
$$

Using integration by parts, we can rewrite the second term as

$$
\begin{aligned}
&\int_{\overline{w}}^{1}n(1-F(w))\phi(w)d((1-\alpha)F(w)+\alpha)^{n-1}\\
=&-n(1-F(\overline{w}))\phi(\overline{w})((1-\alpha)F(\overline{w})+\alpha)^{n-1}\\
&\qquad-n\int_{\overline{w}}^{1}((1-\alpha)F(w)+\alpha)^{n-1}(-\phi(w)f(w)+(1-F(w))\phi'(w))dw.
\end{aligned}
$$

Using integration by parts and (21), we can rewrite the third term as

$$
\begin{aligned}
&-\int_r^{\underline{w}} n(F(w)-F(\xi))\tilde{\phi}(w)d((1-\alpha)F(w)+\alpha F(\xi))^{n-1} \\
=&-n\phi(\overline{w})(F(\underline{w})-F(\xi))((1-\alpha)F(\underline{w})+\alpha F(\xi))^{n-1} \\
&\quad + n\int_r^{\underline{w}}((1-\alpha)F(w)+\alpha F(\xi))^{n-1}(\tilde{\phi}(w)f(w)+(F(w)-F(\xi))\tilde{\phi}'(w))dw,
\end{aligned}
$$

where

$$
\tilde{\phi}(w)=\phi(w)+\frac{(\phi(\overline{w})-\phi(\underline{w}))f(\underline{w})}{f(w)},
$$

as defined in step (iii) of the proof of Theorem 2, and is shown there to be strictly increasing for $w \in (r, \underline{w})$.

Now we can rewrite the partial derivatives of the Lagrangian (22) with respect to $\alpha$ as

$$
\begin{aligned}
&-n\phi(\overline{w})\chi(1-F(\overline{w})+F(\underline{w})-F(\xi)) \\
&\quad + n\int_r^{\underline{w}}((1-\alpha)F(w)+\alpha F(\xi))^{n-1}((\phi(\overline{w})-\phi(\underline{w}))f(\underline{w})+(F(w)-F(\xi))\tilde{\phi}'(w))dw \\
&\quad - n\chi\left(\int_{\underline{w}}^{\overline{w}}\phi(w)f(w)dw-\pi(t)\right)-n\int_{\overline{w}}^{1}((1-\alpha)F(w)+\alpha)^{n-1}(1-F(w))\phi'(w)dw \\
<&-n\phi(\overline{w})\chi(1-F(\overline{w})+F(\underline{w})-F(\xi)) \\
&\quad + n\chi(\underline{w}-t)(\phi(\overline{w})-\phi(\underline{w}))f(\underline{w})+n\chi\int_r^{\underline{w}}(F(w)-F(\xi))\tilde{\phi}'(w)dw \\
&\quad - n\chi\left(\int_{\underline{w}}^{\overline{w}}\phi(w)f(w)dw-\pi(t)\right)-n\chi\int_{\overline{w}}^{1}(1-F(w))\phi'(w)dw \\
=&-n\chi\Big(\phi(\overline{w})(1-F(\overline{w})+F(\underline{w})-F(\xi))-(\underline{w}-t)(\phi(\overline{w})-\phi(\underline{w}))f(\underline{w}) \\
&\quad -(F(\underline{w})-F(\xi))\phi(\overline{w})+(\underline{w}-r)(\phi(\overline{w})-\phi(\underline{w}))f(\underline{w})+\pi(r)-\pi(\underline{w}) \\
&\quad +\pi(\underline{w})-\pi(\overline{w})-\pi(t)-(1-F(\overline{w})\phi(\overline{w})+\pi(\overline{w})\Big) \\
=&-n\chi(\pi(r)+(t-r)\pi'(r)-\pi(t)) \\
<&0,
\end{aligned}
$$

where the first line follows from the binding incentive condition (11) for the informed buyer with valuation $\underline{w}$, together with $\tilde{\phi}'(w) > 0$ for all $w \in (r, \underline{w})$, $\phi'(w) > 0$ for all $w > \overline{w}$, and

$$((1-\alpha)F(\underline{w}) + \alpha F(\xi))^{n-1} < \chi < ((1-\alpha)F(\overline{w}) + \alpha)^{n-1},$$

the second line follows from integration by parts, together with (13), the third line follows from (13) again, and the last line follows the strict concavity of $\pi(\cdot)$.

For comparative statics with respect to $\xi$, we take partial derivatives of the Lagrangian (22) with respect to $\xi$ to get

$$\begin{aligned}&\frac{\partial \chi}{\partial \xi}\left(n(1-\alpha)\int_{\underline{w}}^{\overline{w}} \phi(w)f(w)dw + n\alpha\pi(t) - \Lambda(\underline{w}-t)\right)\\ &+\int_r^{\underline{w}} (n-1)\alpha f(\xi)((1-\alpha)F(w)+\alpha F(\xi))^{n-2}(n(1-\alpha)\phi(w)f(w)+\Lambda)dw.\end{aligned}$$

Using the formula in Lemma 3 and taking derivatives, we have

$$\frac{\partial \chi}{\partial \xi} = \frac{\alpha f(\xi)\left(\chi - ((1-\alpha)F(\underline{w}) + \alpha F(\xi))^{n-1}\right)}{(1-\alpha)(F(\overline{w}) - F(\underline{w})) + \alpha(1-F(\xi))} > 0.$$

Combining the expression for $\Lambda$ with the first order condition (12), we have

$$n(1-\alpha)\int_{\underline{w}}^{\overline{w}} \phi(w)f(w)dw + n\alpha\pi(t) - \Lambda(\underline{w}-t) = n((1-\alpha)(F(\overline{w}) - F(\underline{w})) + \alpha(1-F(\xi)))\phi(\overline{w}),$$

which is strictly positive. Thus the first term in the derivative of the Lagrangian with respect to $\xi$ is strictly positive. The second term is clearly positive. The proposition follows immediately from the Envelope Theorem.

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